DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
JANUARY 2016

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correct.

PhD: 75% with at least three substantially correct.

1. Consider the fixed point iteration $x_{n+1} = g(x_n)$ with the function $g(x) = Cx(1 - x^2)$.
   (a) Find all fixed points, as a function of C.
   (b) For what values of C does the iteration converge locally to the various fixed points (i.e., converge when the initial guess is sufficiently close)?

2. Find coefficients a_0, a_1, a_2 and a_3 and the node x_0 such that

$$\int_{-1}^{1} f(x)dx \approx a_0 f(-1) + a_1 f(-x_0) + a_2 f(x_0) + a_3 f(1)$$

   has the highest degree of accuracy.

3. Let $A = \begin{bmatrix} 10^{-20} & 2 \\ 1 & 3 \end{bmatrix}$.
   (a) Compute the LU decomposition of A in exact arithmetic.
   (b) Compute the LU decomposition in finite precision floating-point arithmetic, assuming 15 decimal digits of accuracy. (Namely, at this precision $1 \oplus 10^{-16} = 1$, but $10^{-16} \neq 0$.)
   (c) Compare the two results.

4. Given an ODE initial-value problem $y' = f(x, y)$ with $y(x_0) = y_0$. Suppose that $x_n = nh, n = 0, ..., N$ where $h = x_n - x_{n-1}$ for all n, and the following scheme is applied to solve the equation:

$$y_{n+1} = y_n + h[b_1 f(x_n, y_n) + b_2 f(x_{n-1}, y_{n-1}) + b_3 f(x_{n-2}, y_{n-2})].$$

   (a) Find b_1, b_2, b_3 for the above method which gives the highest order local truncation error.
   (b) Give the local truncation error of the method obtained.
   (c) State the order of the method obtained.

5. Suppose that equispaced points (including the two boundary points) are used in the polynomial interpolation of the step function

$$f(x) = \begin{cases} 1, & x \in [-1, 0], \\ -1, & x \in (0, 1]. \end{cases}$$

   (a) Find the interpolation polynomials of degree one and two (denoted as $p_1(x)$ and $p_2(x)$))
   (b) Compute the L_∞ (or *maximum*) errors of these three interpolations.
   (c) Is the polynomial interpolation on equispaced points convergent for $f(x)$ in terms of the *maximum* error? Why or why not?

6. Again for the step function

$$f(x) = \begin{cases} 1, & x \in [-1, 0], \\ -1, & x \in (0, 1], \end{cases}$$

solve the following minimization problem

$$\min_{a_0, a_1, a_2} \int_{-1}^{1} \left(f(x) - \sum_{k=0}^{2} a_k P_k(x) \right)^2 dx,$$

and find the minimum. Here $\{a_k\}$ are constants, and $\{P_k(x)\}$ are the Legendre polynomials which are orthogonal on $[-1, 1]$:

$$\int_{-1}^{1} P_k(x) P_l(x) dx = \frac{2}{2k+1} \delta_{kl}.$$

The Legendre polynomials can be obtained by the following recurrence relation:

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad (m+1)P_{m+1}(x) - (2m+1)xP_m(x) + mP_{m-1}(x) = 0.$$

7. **Definition:** An $n \times n$ complex matrix A is said to be *normal* if it commutes with its conjugate transpose, i.e. $AA^* = A^*A$, where $A^* = \bar{A}^T$.

   (a) Use Schur's theorem to prove the following *Spectral Theorem for Normal Matrices*:

   $$A \text{ is normal} \iff A \text{ is unitarily similar to a diagonal matrix}$$

   (b) Given *any* square matrix A, it can be shown that

   $$\lim_{k \to \infty} A^k = 0 \iff \rho(A) < 1,$$

   where $\rho(A)$ is the *spectral radius* of A. Prove this result when A is normal.