Complex analysis qualifying exam

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Do 8 out of the following 10 questions.

Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points with 3 questions essentially correct. To pass at the PhD level it is sufficient to have 55 points with 4 questions essentially correct.

Note: All answers should be justified carefully.

(1) Let C be a square with vertices $(\pm 4, \pm 4)$ oriented counter-clockwise. Evaluate the integral

$$\int_C \frac{z^5 e^{1/z}}{1 - z^4} dz$$

(2) Find *all* holomorphic functions $f(z)$ which map the open set

$$\{z \,:\, -\pi < \mathrm{Re} z < \pi\}$$

one-to-one onto the open unit disk

$$\{z \,:\, |z| < 1\}$$

and such that $f(0) = 0$. Justify your answer!

(3) Let $p_1, p_2, p_3, p_4 \in \mathbb{C}$ be 4 arbitrary distinct points. Let C (resp. D) be a unique circle or line passing through points p_1, p_2, p_3 (resp. p_1, p_3, p_4). Show that C and D are perpendicular at p_1 if and only if the cross-ratio (p_1, p_2, p_3, p_4) is a purely imaginary number.

(4) Let $f(z) = (z-\alpha_1)\dots(z-\alpha_n)$ be a complex polynomial with derivative $f'(z) = n(z-\beta_1)\dots(z-\beta_{n-1})$.

(a) Show that if $\operatorname{Re}\alpha_i > 0$ for every i then $\operatorname{Re}\beta_i > 0$ for every i (Hint: consider $f'(\beta_i)/f(\beta_i)$).

(b) Show that if $|\alpha_i| < 1$ for every i then $|\beta_i| < 1$ for every i.

(5) Show that the series

$$f(z) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 + 2\pi i n z}$$

defines an entire function.

(6) Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2}\,dx$$

for any positive real number a.

(7) Let λ be a real number larger than 1. Show that the equation

$$\lambda - z - e^{-z} = 0$$

has precisely one solution in the half plane $\{z : \operatorname{Re}(z) > 0\}$. Moreover, the solution is real.

(8) Let f be a non-constant entire function. Show that the image of f is dense in the complex plane.

(9) Evaluate the integral

$$\int_0^{2\pi} \frac{1}{2-\sin(\theta)}\,d\theta.$$

(10) Consider the Laurent series $\tan(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, which is valid in the annulus $\frac{\pi}{2} < |z| < \frac{3\pi}{2}$. Find the coefficients a_n with index $-\infty < n \leq -1$. Hint: Use the integral formula for the coefficients.