DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
MASTER'S OPTION EXAM — APPLIED MATH
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Do 5 of the following questions. Each question carries the same weight. Passing level is 60% and at least two questions substantially correct.

1. [20 points] Consider the weakly perturbed oscillator:

$$\ddot{x} + x = -2\epsilon\dot{x}$$

(a) Solve the system exactly with initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$.
(b) Attempt to solve it by regular perturbation theory $x(t) = x_0(t) + \epsilon x_1(t) + \dots$ and obtain the first two orders ($x_0(t)$ and $x_1(t)$). Explain the problem arising.
(c) Try to solve for the leading order x_0, by the same perturbative expansion, using *also* two-timing $\tau = t$ and $T = \epsilon t$. Does that give a better approximation and why ?

2. [20 points] Consider the following limit cycle problems:
(a) Estimate for $\mu \gg 1$ the period of the limit cycle of

$$\ddot{x} + \mu(x^2 - 4)\dot{x} + x = 1$$

(b) Show that there are no limit cycles for the two-dimensional dynamical system:

$$\begin{aligned} \dot{x} &= x^3 - 2 - 2xe^{x^2+y^2} \\ \dot{y} &= y - 2ye^{x^2+y^2} \end{aligned}$$

3. [20 points] Consider the problem $u_{tt} = c^2 u_{xx}$ with homogeneous Neumann boundary conditions in $(0, \pi)$ and initial conditions $u(x, 0) = x$ and $u_t(x, 0) = 0$.

(a) Solve the PDE by separating the variables, applying the boundary conditions and then the initial condition.
(b) Apply Parseval's identity for the Fourier series decomposition of $f(x) = x$ that you computed in (a) to obtain that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$$

4. [20 points] Consider the spherical wave equation in 3 dimensions of the form:

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right)$$

(a) Change variables to $v = ru$ to get to a PDE for v.
(b) Use the initial conditions $u(r, 0) = \phi(r)$ and $u_t(r, 0) = \psi(r)$, taking both ϕ and ψ to be even functions of r, to derive the D'Alembert solution for the above spherical wave equation.
5. [20 points] Consider the diffusion equations

$$u_t - ku_{xx} = f \quad \text{and} \quad v_t - kv_{xx} = g\,,$$

with

$$u \leq v \quad \text{for} \quad x = 0\,,\, x = l\,,\, t = 0\,.$$

(a) Use an argument based of the maximum principle, to establish that $u \leq v$ for all $0 \leq x \leq l$ and $0 \leq t$, when $f = g$.
(b) Can you modify your argument to prove the same result when $f \leq g$ for all their values?
(c) Assume $v_t - v_{xx} \geq \sin(x)$ for $0 \leq x \leq \pi$ and $t \geq 0$. Furthermore, assume that $v(0, t) \geq 0$, $v(t, \pi) \geq 0$ and $v(x, 0) \geq 0$. Then, using (a-b) show that

$$v(x, t) \geq (1 - e^{-t}) \sin(x)\,.$$

6.[20 points] (a) Draw the bifurcation diagram of the differential equation

$$\frac{dy}{dt} = y^4 + \mu y^2$$

where the parameter μ can take any positive or negative values, as well as 0.

(b) Sketch qualitatively some solutions of the differential equation in all cases of interest (you need to identify them first!).

7. Consider the system

$$x' = y\,, \quad y' = -by + x - x^3\,. \tag{2}$$

(a) Show that this system is dissipative, if $b > 0$.

(b) Draw the phase portrait of the system when $b > 0$; justify your answer.