BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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- Do 7 of the following 9 problems. **Show your work!**
- Passing Standard:
  - Master's level: 60% with three questions essentially complete (including at least one from each part)
  - Ph. D. level: 75% with two questions from each part essentially complete

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Part I. Linear Algebra

1. Find two 2×2 matrices A, B with $A \neq \pm B$ such that

$$A^2 = B^2 = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}.$$

Note: It is possible to solve this problem *without* doing excessive, brute force computation.

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2. Let A be a real, square matrix. Suppose $A = -A^t$.
(a) Show that if A is 3×3 then $\det(A) = 0$.
(b) Show that if A is 4×4 and invertible then $\det(A)$ is positive.

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3. Let V be the vector space of real polynomials of degree ≤ 2. Define an inner proeduct on V by

$$\langle p, q \rangle = \frac{1}{2} \int_{-1}^{1} p(x)q(x)dx.$$

(a) Find an orthonormal basis for V consisting of polynomials of degree $0, 1$ and 2, respectively.
(b) Find the second degree polynoimal that solves the minimization problem

$$\min_{p \in V} \int_{-1}^{1} (p(x) - x^3)^2 dx.$$

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4. Two $n \times n$ real matrices A, B are said to be *similar* if there is an invertible matrix M such that

$$A = MBM^{-1}.$$

Describe all equivalence classes of 2×2 matrices.

Part II. Advanced Calculus

1. Find the highest point of intersection of the sphere $x^2 + y^2 + z^2 = 30$ and the cone $x^2 + 2y^2 - z^2 = 0$.

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2. Let $D \subset \mathbf{R}^n$ be a closed *unbounded* subset, and let $f : D \to \mathbf{R}^m$ be a continuous function. Suppose $f(x) \to 0$ as $||x|| \to \infty$ ($x \in D$). Prove that f is uniformly continuous on D.

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3. Let S be a closed, oriented surface in $\mathbf{R}^3$ with unit normal vector $\vec{n}$. Let $\vec{c} \in \mathbf{R}^3$ be a *constant* vector field. Show that the surface integral

$$\iint_S \vec{c} \bullet \vec{n} \, dS$$

is identically zero.

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4. Find domain of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(n+x)^n}{n^{n+x}}$.

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5. Let f be a strictly increasing, continuous function on the closed interval $[a, b]$. Then there is a uniquely defined inverse function g defined on the interval $[f(a), f(b)]$, i.e. $g(f(x)) = x$ for all $x \in [a, b]$ (you do not have to prove this).

Prove that g is continuous.
