NAME:

Advanced Analysis Qualifying Examination
Department of Mathematics and Statistics
University of Massachusetts

Monday, January 11, 2016

Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question and on the blank page after each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.

2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space with $\mu(X) > 0$ and let f be a nonnegative, Borel-measurable function mapping X into $[0, \infty)$. Assume that $f \not\equiv 0$ a.e. For $A \in \mathcal{M}$, define

$$\nu(A) = \int_A f \, d\mu \, .$$

(a) Prove that there exists $A \in \mathcal{M}$ such that $\nu(A) > 0$. This proves that ν is nontrivial.

(b) Prove that ν is a measure on $\mathcal{M}$ and that $\nu \ll \mu$.

(c) Let ψ be any nonnegative, Borel-measurable function mapping X into $[0, \infty)$. Prove that

$$\int_X \psi \, d\nu = \int_X \psi \, f \, d\mu \, .$$

(d) In the particular case where f is also a bounded function, prove that ν is a *finite* measure.

2. In each case compute $\int_X g\,d\rho$, where $X = \{1, 2, 3, \ldots\} = \mathbb{N}$ and all subsets of X are measurable.

   (a) $g(x) = (1/5)^x$ and ρ is counting measure; i.e., $\rho(A)$ equals the number of elements, finite or infinite, in A.

   (b) $g(x) = 1/x$ and ρ is counting measure.

   (c) $g(x) = (x8^x)^{-1}$ and $\rho(\{k\}) = k$ for all $k \in X$.

   (d) $g(x) = e^{x/7}$ and $\rho(A) = 1_A(7)$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f and $\{f_n, n \in \mathbb{N}\}$ be measurable functions on X.

(a) Define what it means for f_n to converge to f in measure.

(b) Suppose that $\mu(X) < \infty$. Prove that $f_n \to f$ a.e implies $f_n \to f$ in measure.

(c) Prove that the converse of (b) is false even for $\mu(X) < \infty$.

Hint. Let $X = [0, 1]$. For $n \in \mathbb{N}$, $i = 1, 2, \ldots, n$, and $x \in [0, 1]$ define

$$A_n^i = \left[\frac{i-1}{n}, \frac{i}{n}\right] \text{ and } f_n^i(x) = 1_{A_n^i}(x).$$

To construct a counterexample to the converse of (b) consider the sequence

$$\{f_1^1, f_2^1, f_2^2, f_3^1, f_3^2, f_3^3, \ldots, f_n^1, f_n^2, \ldots, f_n^n, \ldots\}.$$

4. Let f be a function mapping a bounded interval $[a, b]$ into $\mathbb{R}$.

(a) Define the concept that f is of bounded variation on $[a, b]$.

(b) Define the concept that f is absolutely continuous on $[a, b]$.

(c) Assume that the function f mapping $[a, b]$ into $\mathbb{R}$ is integrable. For $x \in [a, b]$ define

$$H(x) = \int_a^x f(t)\, dm(t).$$

Using only definitions, prove that H is continuous on $[a, b]$ and is of bounded variation on $[a, b]$.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{f_n, n \in \mathbb{N}\}$ be a sequence of functions in $L^1(X)$ that converges in measure to another function $f \in L^1(X)$. Let $\gamma(x) := \sup_n |f_n(x)|$ for $x \in X$.

   (a) Prove that if $\gamma \in L^1(X)$, then f_n converges to f in $L^1(X)$.

   Hint. Start by considering an arbitrary subsequence f_{n_j} of f_n.

   (b) Give an example to show that if γ is **not** in $L^1(X)$, then f_n need not converge to f in $L^1(X)$.

6. (a) Let $\mathcal{H}$ be an infinite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$. Prove that every orthonormal sequence in $\mathcal{H}$ converges weakly to 0. That is, show that for any orthonormal sequence $\{e_n, n \in \mathbb{N}\} \subset \mathcal{H}$, we have $\langle e_n, w \rangle \to 0$ as $n \to \infty$ for all $w \in \mathcal{H}$.

   (b) Let $f_n(x) := \sqrt{2}\cos(2\pi n x)$ for $x \in [0,1]$. Prove that $f_n \to 0$ weakly in $L^2([0,1])$ but that f_n does not converge to 0 in measure with respect to Lebesgue measure m on $[0,1]$.

   (c) Let $f_n(x) := n\, 1_{[0,\frac{1}{n}]}(x)$ for $x \in [0,1]$. Prove that $f_n \to 0$ m-a.e. on $[0,1]$ but that f_n does not converge to 0 weakly in $L^2([0,1])$.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of measurable subsets of X.

(a) Define $\limsup_{n\to\infty} C_n$ and $\liminf_{n\to\infty} C_n$ in terms of appropriate unions and intersections.

(b) Prove that $x \in \liminf_{n\to\infty} C_n$ if and only if x lies in all but finitely many C_n and that $x \in \limsup_{n\to\infty} C_n$ if and only if x lies in infinitely many C_n.

(c) Prove that $\mu(\liminf_{n\to\infty} C_n) \leq \liminf_{n\to\infty} \mu(C_n)$.

(d) Prove that if $\sum_{n=1}^{\infty} \mu(C_n) < \infty$, then $\mu\left(\limsup_{n\to\infty} C_n\right) = 0$.

8. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. For $\alpha \in \mathbb{R}$ define $\lambda(\alpha) = \mu\{|f| > \alpha\}$.

(a) Prove that for $f \in L^1(d\mu)$

$$\int_X |f| d\mu = \int_0^\infty \lambda(\alpha) d\alpha.$$

(b) Prove that for $f \in L^p(d\mu)$ for $1 < p < \infty$

$$\int_X |f|^p d\mu = p \int_0^\infty \alpha^{p-1} \lambda(\alpha) d\alpha.$$

Hint. For parts (a) and (b) verify the respective equations

$$\int_X |f| d\mu = \int_X \left(\int_0^{|f|} d\alpha \right) d\mu \text{ and } |f|^p = \frac{1}{p} \int_0^{|f|} \alpha^{p-1} d\alpha$$

and then use these equations to prove the respective parts (a) and (b).