DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - PROBABILITY
January 2015

Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Suppose that the random variable X has a Poisson distribution with mean μ. The probability mass function is:

$$f(x|\mu) = \frac{e^{-\mu}\mu^x}{x!}, \quad \text{for } \mu > 0; \quad x = 0, 1, 2, \cdots.$$

It is known that

$$e^z = \sum_{k=0}^{\infty} \left(\frac{z^k}{k!} \right).$$

   (a) (7pts) Find the moment-generating function for X.

   (b) (7pts) If $X_1, X_2, \cdots, X_n$ are independent Poisson random variables with means $\mu_1, \mu_2, \cdots, \mu_n$, find the moment generating function of $Y = \sum_{i=1}^{n} X_i$.

   (c) (7pts) What is the distribution of Y?

2. Of the travelers arriving at a small airport, 60% fly on major airlines, 30% fly on privately owned planes, and the remainder fly on commercially owned planes not belonging to a major airline. The conditional probability of persons traveling on business, given that they fly on major airlines, is 40%. The conditional probability of persons traveling on business, given that they fly on privately owned planes, is 50%. The conditional probability of persons traveling on business, given that they fly on commercially owned planes not belonging to a major airline, is 80%.

   (a) (7pts) Suppose that we randomly select one person arriving at this airport. What is the probability that the person is traveling on business?

   (b) (7pts) What is the probability that the randomly selected person in (a) arrived on a major airline, given that the person is traveling for business reasons?

   (c) (7pts) n travelers are randomly selected and surveyed. Let X_1 be the number of persons among the n surveyed that travel for business reasons and on major airlines, X_2 be the number of persons among the n surveyed that travel for non-business reasons and on major airlines, X_3 be the number of persons among the n surveyed that travel for business reasons and on privately owned planes, X_4 be the number of persons among the n surveyed that travel for non-business

reasons and on privately owned planes, X_5 be the number of persons among the n surveyed that travel for business reasons and commercially owned planes not belonging to a major airline, and X_6 be the rest. What is the joint probability distribution of X_1, X_2, X_3, X_4, X_5, and X_6? Give the name and the parameters of the distribution.

(d) (7pts) What is the distribution of $Y = (Y_1, Y_2, Y_3)$ where $Y_1 = X_1 + X_2$, $Y_2 = X_3 + X_4$, and $Y_3 = X_4 + X_5$.

(e) (7pts) What is the conditional distribution of X_1 given $X_1 + X_3 + X_5 = r$ for some $0 < r < n$?

(f) (7pts) Find the correlation coefficient between Y_1 and Y_2.

3. Suppose X is a random variable with distribution given by

$$C \sim Bernoulli(p)$$
$$X|C \sim N(\mu_C, \sigma_C^2)$$

(a) (7pts) Write down the probability density function of X.

(b) (7pts) Find $E(X)$.

(c) (7pts) Find $Var(X)$.

4. (8pts) Let $X_1, \cdots, X_n$ be iid $N(0,1), n \geq 1$. We denote

$$U_n = \frac{\sqrt{n}\{X_1 + X_2 + \cdots + X_n\}}{X_1^2 + X_2^2 + \cdots + X_n^2}.$$

Show that U_n converges to the standard Normal distribution as n goes to infinity. {*Hint: Use both CLT and Slutsky's Theorem.*}

5. (8pts) Let $X_1, \cdots, X_n$ be iid Bernoulli(p) with $0 < p < 1$. Let $V_n = n^{-1} \sum_{i=1}^n X_i$. Show that

$$\frac{\sqrt{n}(V_n^{-1} - p^{-1})}{\sqrt{(1-p)p^{-3}}} \xrightarrow{D} N(0,1) \text{ as } n \longrightarrow \infty.$$