# COMPLEX ANALYSIS BASIC EXAM
# UNIVERSITY OF MASSACHUSETTS, AMHERST
# DEPARTMENT OF MATHEMATICS AND STATISTICS
# JANUARY 2015

- Each problem is worth 10 points.
- Passing Standard: **Do 8 of the following 10 problems**, *and*
  - Master's level: 45 points with three questions essentially complete
  - Ph. D. level: 55 points with four questions essentially complete
- Justify your reasoning!

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1.

(a) Find the Taylor series expansion of the function $f(z) = \sin^2(z)$ around the origin. For what values of $z$ does the series converge?

(b) Find the Laurent series expansion of the function $f(z) = \dfrac{1}{z^2 - 3z + 2}$ valid near (and centered at) the point $z_0 = 1$. For what values of $z$ does the series converge?

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2. Let $f(z)$ be analytic inside and on the circle $|z| = R > 0$. Suppose $|f(z)| < M$ along $C$.

(a) Give the precise statement of the Cauchy inequality for the $n$-th derivative $f^{(n)}(x)$ of $f$ $(n \geq 1)$.

(b) Give examples to show that Cauchy inequality is the best possible, i.e. the estimate growth of $f^{(n)}(x)$ cannot be improved for all functions analytic inside and on $|C| = R$.

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3. Evaluate the integral $\displaystyle\int_C z^n e^{2/z} dz$, where $n$ is an integer and $C$ is the circle $|z| = R > 0$, positively oriented. Justify your reasoning.

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4(a) Let $f$ be a non-constant analytic function on a connected open set $U$. If $f$ does not vanish anywhere on $U$, show that $|f(z)|$ cannot attain a minimum value inside $U$.

(b) Let $f$ be an anaytic function on the open unit disk $D$. Suppose $f$ sends $D$ bijectively and conformally onto a connected open set containing $D$ and with $f(0) = 0$, show that $|f'(0)| \geq 1$, with equality precisely for $f(z) = cz$ with $|c| = 1$.

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5. Compute the integral

$$\int_0^\infty \frac{\cos(x)}{(x^2+1)^2} dx$$

Justify all your steps.

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6. Let $f(z)$ be meromorphic but not holomorphic on $\mathbf{C}$. Show that $g(z) := e^{f(z)}$ is not meromorphic on $\mathbf{C}$.

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7. Prove or disprove the following statements:
   (1) The function
   $$f(z) \;=\; \frac{1}{(z-1)(z-2)}$$
   has an anti-derivative in the annulus $\{z \in \mathbb{C} \;:\; |z| > 2\}$.
   (2) The image of the complex plane $\mathbb{C}$ under a non-constant entire function is dense in $\mathbb{C}$.
   (3) If $f$ is holomorphic on $\Delta^* \;=\; \{z \in \mathbb{C} \;:\; 0 < |z| < 1\}$ and $\mathrm{Res}_0(f) = 0$, then $f$ has a removable singularity at 0.

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8. Determine the number of solutions of $\cos(z) = cz^n$ in the open unit disk for every positive integer $n$ and constant $c$ satisfying $|c| > e$.

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9. Find all fractional linear transformations that takes the circle $|z| = 1$ to the real axis.

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10. Let $f_n$ be a sequence of holomorphic functions converging uniformly to a function $g$ on some connected open set $U$. Suppose that $g$ vanishes at some $z_0 \in U$ but is not identically zero on $U$. Prove that $z_0$ is an accumulation point of zeroes of the functions $f_n$. Is the assertion true for real analytic functions?

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