ADVANCED CALCULUS/LINEAR ALGEBRA BASIC EXAM

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JANUARY 2015

Complete 7 of the following 9 problems. Please show your work. The passing standards are:

- Master's level: 60% with three questions essentially correct (including one from each part);
- Ph.D. level: 75% with two questions from each part essentially correct.

Linear Algebra

(1) The only eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 0 & 5 & 0 \\ 0 & 5 & 0 & 5 \\ 0 & 5 & 0 & 5 \\ 5 & 0 & 5 & 0 \end{pmatrix}$$

are 0 and 10. Find, with a minumum of computation, the rank, nullity, trace, determinant, and bases of the eigenspaces of A. Is A diagonalizable?

(2) Let A, B be $n \times n$ matrices over a field k.
  (a) Show that $\operatorname{rank} AB \leq \min(\operatorname{rank} A, \operatorname{rank} B)$.
  (b) Show that if $\operatorname{rank} AC = \operatorname{rank} C$ for all $n \times n$ matrices C, then A is invertible.

(3) Let A be a (real) orthogonal $n \times n$ matrix, so $A^t A = I$ (where A^t denotes the transpose matrix).
  (a) Prove that $\det A = \pm 1$.
  (b) Prove that x and Ax have the same length, for all $x \in \mathbb{R}^n$.
  (c) Prove that all eigenvalues λ of A satisfy $|\lambda| = 1$.
  (d) If $n = 3$ and $\det A = 1$, prove that 1 is an eigenvalue of A.

(4) (a) If $V \subset W$ are subspaces of $\mathbb{R}^n$, and $\dim W = \dim V + 1$, show that $V^\perp \cap W$ is one-dimensional, where $V^\perp$ is defined with respect to the usual inner product on $\mathbb{R}^n$.
  (b) Given subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{R}^n$$

with $\dim V_i = i$ for all i, show that there exists an orthogonal matrix A such that the span of the first i columns is V_i. How many such matrices are there?

Advanced Calculus

(5) Find the average value of the function $F(x) = \int_x^1 \sin(t^2)dt$ on $[0, 1]$.

(6) (a) Let $\mathbf{F}$ be a continuously differentiable vector field and g a continuously differentiable function on a domain in $\mathbb{R}^3$. Prove the following relation involving the divergence and the gradient:

$$\mathrm{div}(g\mathbf{F}) = \langle \mathrm{grad}(g), \mathbf{F}\rangle + g\,\mathrm{div}(\mathbf{F}).$$

(b) Show that there is exactly one value of n so that

$$\mathbf{F}(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||^n},$$

which is defined everywhere on $\mathbb{R}^3$ except at the origin, is divergence-free.

(c) For the $\mathbf{F}$ you found in (b) above, compute the flux integral

$$\int_S \mathbf{F} \cdot d\mathbf{A},$$

where S is the sphere $(x-1)^2 + y^2 + z^2 = 100$. (Hint: use the divergence theorem, a.k.a. Gauss's theorem.)

(7) (a) Determine the radius of convergence R and the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n} x^n.$$

(b) If a power series $\sum a_n x^n$ has radius of convergence $R > 0$ and if $0 < R' < R$, prove that the series converges uniformly on $[-R', R']$. Conclude that the series defines a continuous function on $(-R', R')$.

(8) Show that the improper Riemann integral $\int_\pi^\infty \frac{\sin x}{x} dx$ converges.

(9) Let a_{ij}, $1 \leq i, j \leq \infty$ be real numbers.

(a) Show that if all $a_{ij} \geq 0$, then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

In particular, this means that the left sum converges if and only if the right one does.

(b) Give an example to show that this need not be true if the assumption $a_{ij} \geq 0$ is dropped.