Department of Mathematics and Statistics
University of Massachusetts

ADVANCED EXAM — DIFFERENTIAL EQUATIONS
JANUARY 2015

Do five of the following seven problems. All problems carry equal weight. Passing level: 75% with at least three substantially complete solutions.

1. Consider the two-dimensional dynamical system

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -x^3\,, \end{aligned}$$

   (a) Show that the linearization of this system at the equilibrium point $(x^*, y^*) = (0, 0)$ is unstable.

   (b) Use a Lyapunov function argument to show that the nonlinear system itself is stable.

   (c) Discuss why these two facts are not contradictory.

2. (a) Exhibit a two-dimensional smooth dynamical system,

$$\begin{aligned} \dot{x} &= f(x, y)\,, \\ \dot{y} &= g(x, y)\,, \end{aligned}$$

   for which the ellipse $x^2/a^2 + y^2/b^2 = 1$ is a limit cycle; a and b are arbitrary (positive) semi-axes.

   (b) Exhibit a three-dimensional dynamical system,

$$\begin{aligned} \dot{x} &= f(x, y, z)\,, \\ \dot{y} &= g(x, y, z)\,, \\ \dot{z} &= h(x, y, z)\,, \end{aligned}$$

   for which the space curve $x^2 + y^2 = 1,\ x + y + z = 1$ is an attractor.

3. Consider an ODE system written in the form

$$\dot{x} = Ax + g(x), \qquad \text{for } x \in \mathbb{R},$$

where $g(x)$ is a smooth function and $|g(x)| = O(|x|^2)$ as $|x| \to 0$. Suppose that the coefficient matrix A has one strictly positive (real) eigenvalue. Prove that the origin $x^* = 0$ is unstable.

4. Consider a harmonic function $u(x, y)$ on the rectangular domain, $R_L = \{ (x, y) \in \mathbb{R}^2 : 0 < x < L, \ 0 < y < 1 \}$, where L is the length.

   (a) For fixed $L > 0$, find the explicit solution $u(x, y)$ having the boundary conditions: $u = 0$ on the sides with $y = 0, y = 1, x = 0$, and $u(L, y) = b \sin \pi y$ on the side with $x = L$; b is any positive constant.

   (b) Now consider a sequence of rectangles R_{L_n} with $L_n \to +\infty$, and allow $b = b(L)$ to depend on L. What growth condition on $b(L)$ is needed to ensure that the corresponding solution sequence, $u_n(x, y)$, tends to zero pointwise as $L_n \to +\infty$?

   (c) Construct a harmonic function $v(x, y)$ on the semi-infinite domain, $R_\infty = \{ (x, y) \in \mathbb{R}^2 : 0 < x < \infty, \ 0 < y < 1 \}$, such that $v = 0$ on the boundary of R_∞, and yet v is positive throughout the interior of R_∞.

5. Consider the elliptic boundary-value problem

$$\begin{aligned} \triangle u + \alpha u &= f(x) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

on a smooth bounded domain Ω in R^n, where $\alpha \in \mathbb{R}$ is a constant, and $f \in L^2(\Omega)$.

   (a) Suppose that $\alpha < \lambda_1(\Omega)$, the smallest eigenvalue of $-\triangle$ on Ω. Show that this boundary-value problem has a unique weak solution $u \in H_0^1(\Omega)$.

   (b) Suppose instead that $\alpha = \lambda_1(\Omega)$. What condition on f is required to ensure the existence of a weak solution? What can be said about the uniqueness of weak solutions in this case?

6. Consider the initial-boundary-value problem:

$$\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 && \text{in} \quad 0 < x < +\infty,\ t > 0 \\
u &= \sin \omega t && \text{on } x = 0,\ t > 0 \\
u &= 0 && \text{for} \ \ 0 < x < +\infty,\ t = 0\,.
\end{aligned}$$

Assume that the classical solution exists and tends to zero as $x \to +\infty$. Note that this heat equation is posed on the semi-infinite line, with an inhomogeneous boundary condition which oscillates with a given frequency ω.

(a) First, ignore the initial condition and explicitly construct a time-periodic solution, $U(x,t)$, that satisfies the PDE and its time-periodic boundary condition.
HINT: Use separation of variables, $U(x,t) = F(x)G(t)$, and allow the separation constant to be *complex*.

(b) Next, show that the solution, $u(x,t)$, of the initial-boundary-value problem itself tends to $U(x,t)$ as $t \to +\infty$.

7. Consider the wave equation with a higher-order damping, namely,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial^3 u}{\partial x^2 \partial t} = 0 \qquad \text{in} \ \ 0 < x < 1, \ \ t > 0\,,$$

along with the free-end boundary conditions: $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(1,t)$. The damping coefficient, γ, is a *positive* constant.

(a) Define a quadratic functional, $E(u)$, that represents the energy associated with such waves, and show that any solution, $u(x,t)$, satisfies

$$\frac{dE}{dt} \leq 0\,.$$

(b) Formulate the initial-value problem for this PDE together with its boundary conditions, and use the inequality proved in (a) to deduce the uniqueness of solutions.