Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

Friday, January 17, 2014

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Let X and Y be spaces, with Y Hausdorff.
  (a) Show that for any continuous function $f\colon X \to Y$, the graph
  $$\Gamma_f = \{(x, f(x)) \mid x \in X\}$$
  is closed in $X \times Y$.
  (b) Show that if $f, g\colon X \to Y$ are continuous and $f|_A = g|_A$ for a dense subset $A \subset X$, then $f = g$.

(2) Let D be the closed disk in $\mathbb{R}^2$, endowed with the topology generated by a basis consisting of the usual open sets, together with all sets of the form
  $$\{p\} \cup B_{1-\epsilon}(\epsilon p),$$
  where $p \in S^1 = \partial D$ and $0 < \epsilon < 1$.
  (a) Describe the topology S^1 inherits as a subspace of D.
  (b) Show that D is not compact.
  (c) Show that D is connected.

(3) Let (X, d) be a bounded metric space. Define a metric on the countable product $\prod_{n=1}^{\infty} X$ by
  $$D((a_n), (b_n)) = \sup_n \frac{d(a_n, b_n)}{n}.$$
  Prove directly from the definitions that the topology induced by D is the same as the product topology.

(4) Let $X \subset \mathbb{R}^2$ be the union of all the line segments joining $(0, 0)$ to $(1/n, 1/n^2)$ for $n = 1, 2, \ldots$.
  (a) Show that X is homeomorphic to the one-point compactification of $(0, 1] \times \mathbb{Z}$.
  (b) Show that X is *not* homeomorphic to the quotient of $[0, 1] \times \mathbb{Z}$ identifying $\{0\} \times \mathbb{Z}$ to a point.

(5) (a) State the Lebesgue number lemma for compact metric spaces.
  (b) Let $f\colon X \to \mathbb{R}$ be a continuous function on a compact metric space. Prove that f is uniformly continuous.

(two problems on back)

(6) Let $\mathcal{C}$ be the set of all continuous functions $[0,1] \to \mathbb{R}$, with the sup metric:

$$d(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|,$$

and let $K \subset [0,1]$ be compact and $U \subset \mathbb{R}$ be open. Show that the set

$$\{f \in \mathcal{C} \mid f(K) \subset U\}$$

is open in $\mathcal{C}$.

(7) Recall that the real projective plane $\mathbb{RP}^2$ is the quotient of S^2 by the equivalence relation generated by $p \sim -p$ for all points $p \in S^2$.

Let $f\colon \mathbb{RP}^2 \to S^1$ be a continuous map. Show that there does not exist a continuous map $g\colon S^1 \to \mathbb{RP}^2$ so that $f \circ g$ is the identity.