BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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- Do 7 of the following 9 problems. **Show your work!**
- Passing Standard:
  - Master's level: 60% with three questions essentially complete (including at least one from each part)
  - Ph. D. level: 75% with two questions from each part essentially complete

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Part I. Linear Algebra

1. Let a, b be real numbers. Determine the rank of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & a^2 & b^2 \end{pmatrix}$.

Note: The answer will depend on the values of a and b; make sure you address *all possible cases.*

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2. Denote by $M_n(\mathbb{C})$ the set of all $n \times n$ complex matrices, and by $\mathbb{C}^n$ the n-dimensional complex vector space. For any $A \in M_n(\mathbb{C})$, define

$$||A|| := \sup_{\vec{x} \in \mathbb{C}^n \setminus \{\vec{0}\}} \frac{|A\vec{x}|}{|\vec{x}|}, \qquad \rho(A) := \max\{|\lambda| : \lambda \in \mathbb{C} \text{ is an eigenvalue of } A\}.$$

(a) Exactly *one* of the following statements is true:

- $\rho(A) \leq ||A||$ for all $A \in M_n(\mathbb{C})$;
- $\rho(A) \geq ||A||$ for all $A \in M_n(\mathbb{C})$.

Determine which statement is true and give a proof (you do not need to give a counter-example for the other statement).

(b) Determine all $A \in M_n(\mathbb{C})$ for which $\rho(A) = ||A||$. *Justify your reasoning.*

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3. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Show that every real matrix B such that $AB = BA$ has the form $sI + tA$ where $s, t \in \mathbb{R}$ and I is the identity matrix.

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4. A square matrix A with complex entries is called *skew-Hermitian* if $A^{\mathrm{T}} = -\overline{A}$. Here, A^{T} is the transpose of A and $\overline{A}$ is the complex conjugate of A.

(a) Suppose A and B are $n \times n$ skew-Hermitian matrices, and a, b are complex numbers. Under what conditions is $C = aA + bB$ a skew-Hermitian matrix?

(b) Show that every eigenvalue of a skew-Hermitian matrix is *purely imaginary*, i.e. the real part is zero. *Justify your reasoning.*

Part II. Advanced Calculus

1. Let $f(x)$ be a differentiable real valued functions on $\mathbb{R}$. Show that all tangent planes to the surface defined by

$$z = xf(y/x) \qquad (\text{for } x \neq 0)$$

intersects at a common point (this common point *could* have $x = 0$).

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2. Let W be the three-dimensional region under the graph of $f(x, y) = e^{x^2+y^2}$ and above the annular region in the xy-plane defined by $1 \leq x^2 + y^2 \leq 2$.
(a) Find the volume of W.
(b) Find the flux of the vector field $\mathbf{F}(x, y, z) = (2x - xy)\vec{i} - y\vec{j} + yz\vec{k}$ out of the region W.

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3. For $x > 0$, define

$$f(x) = \int_0^\infty t^{x-1}e^{-t}dt = \int_0^\infty t^x e^{-t}\frac{dt}{t}.$$

(a) Show that f is continuous.
(b) Use integration by part to show that $f(x + 1) = xf(x)$.

Note: The function f is defined by an improper integral; make sure you address all convergence issues in your argument.

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4. For each positive integer n, define

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n).$$

(a) Show that $a_n > a_{n+1}$ for every $n \geq 1$.
(b) Define $b_n = a_n - (1/n)$. Show that $b_{n+1} > b_n$ for every $n \geq 1$.
(c) Show that the two sequences $\{a_n\}_n$ and $\{b_n\}_n$ are both convergent and that they both converge to the same finite value.

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5. Let $\vec{a}, \vec{b}, \vec{c}$ be three distinct points in $\mathbb{R}^n$. Define a function $\mathbb{R}^n \to \mathbb{R}$ by

$$f(\vec{x}) = |\vec{x} - \vec{a}|^2 + |\vec{x} - \vec{b}|^2 + |\vec{x} - \vec{c}|^2.$$

Find all point(s) $\vec{x} \in \mathbb{R}^n$ at which f reaches its minimum and find the minimum value. *Justify your reasoning.*
