UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Advanced Exam - Probability and Mathematical Statistics
Wednesday, January 15, 2014

Seventy points are required to pass. At least twenty-five must come from problems 1 and 2, and at least twenty five points must come from problems 3-5.

1. *Multivariate Normal Distribution* (25 points)

   (a) Let X be normally distributed with mean μ_x and variance σ_x^2, and let Y be normal with mean μ_y and variance σ_y^2. Prove or disprove the following statement: the vector (X, Y) is multivariate normal.

   (b) Suppose $U|V = v$ is normal with mean $\mu_{u|v}$ and variance $\sigma_{u|v}^2$ and V is normal with mean μ_v and variance σ_v^2. Is the joint distribution of U and V necessarily normal? Why or why not? Is the marginal distribution of U necessarily normal? Why or why not?

   (c) If two random variables are independent, are they necessarily uncorrelated? Why or why not?

   (d) If two random variables are uncorrelated, are they necessarily independent? Why or why not?

   (e) Suppose $U \sim N(0, 1)$, $V \sim N(0, 1)$, and $Cov(U, V) = 0$. Are U and V necessarily independent? Why or why not?

2. *Linear and Quadratic Forms* (25 points) Let $Y \sim N_p(\mu, I)$. Let A and B be $k \times p$ matrices. Suppose that $AB' = 0$.

   (a) Prove that AY and BY are independent.

   (b) Prove that $AB' = 0$ implies that $Y'AY$ and $Y'BY$ are independent.

   (c) Derive the expectation and variance of of $Y'AY$.

3. *Exponential Family* (15 points) Let $\beta = (\beta_1, \ldots, \beta_k)'$ and $T(x)$ be the vector $(T_1(x), \ldots, T_k(x))'$. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables whose probability distribution function is in the exponential family:

$$f(x, \beta) = \exp\left\{\beta' T(x) + b(\beta) + c(x)\right\}.$$

   (a) Give an example with $k > 1$ and show what $\beta, T(x), b(\beta)$, and $c(x)$ are.

   (b) What is the score equation for β? (k is not necessarily 1.)

   (c) For the $k = 1$ case, derive a simple expression for $E\left\{T(X_i)\right\}$.

   (d) For the $k = 1$ case, derive a simple expression for $Var\left\{T(X_i)\right\}$.

(e) For the $k = 1$ case, what are the mean and variance of the score equation?

4. *Probability Definitions* (20 Points)

   (a) Give a precise definition for convergence in distribution

   (b) Give a precise definition for convergence in probability

   (c) Give a precise definition for convergence almost surely

   (d) Give examples of sequences of random variables that converge in distribution but not in probability

   (e) Give examples of sequences of random variables that converge in probability but not almost surely

5. *Law of Large Numbers* (15 points) Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and let $X_i, i = 1, \ldots,$ be a sequence of independent random variables mapping Ω to R. Suppose that $E(X_i) = 0$ and $E(X_i^4) < \infty$ for all i. Let $S_n = \sum_{i=1}^{n} X_i$. Prove that $n^{-1} S_n \rightarrow 1$ with probability 1.