University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
For January 2014

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Let $f\colon M \to N$ be a smooth map and let
$$G = \{(p, f(p))\,;\, p \in M\} \subset M \times N$$
be the graph of f. Show that G is an embedded submanifold.

2. Consider the trivial vector bundle $E = T^2 \times \mathbb{R}^n$ over the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.
   (a) Let $\nabla = d + A\,dx + B\,dy$ be a connection on E where $A, B \in \mathbf{gl}(n, \mathbb{R})$ are $n \times n$ matrices, dx, dy the coordinate differentials on $\mathbb{R}^2$ (which are well defined on T^2), and d is the trivial connection (i.e. directional derivative of $\mathbb{R}^n$ valued functions). Show that such a connection is flat if and only if A and B commute, that is $[A, B] = 0$.
   (b) Show that if A and B have a common k-dimensional kernel, then there exist k linearly independent parallel sections of E.

3. Consider the involution τ on $S^1 \times S^2$, where
$$\tau : (z, x) \mapsto (\bar{z}, -x), \quad z \in S^1 \subset \mathbb{C}, x \in S^2 \subset \mathbb{R}^3.$$
Show that (a) τ is free, (b) $S^1 \times S^2$ is orientable, (c) τ is orientation-preserving.

4. Define vector fields V, W on $\mathbb{R}^2$ by
$$V = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad W = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$
   (a) Show that V, W are not commuting vector fields by computing $[V, W]$.
   (b) Show that the flows θ_t, ψ_s generated by V, W are not commuting by explicit verification.

5. Let g be a Riemannian metric on a smooth n-dimensional manifold M. Define a map $F : TM \to \mathbb{R}$ by sending $X \in T_pM$ to $g_p(X, X) \in \mathbb{R}$. Show that F is smooth and $F^{-1}(1) \subset TM$ is a $(2n-1)$-dimensional embedded submanifold.

6. Consider S^2, the unit sphere in $\mathbb{R}^3$, equipped with a symplectic form ω which is the area form induced from the Euclidean metric on $\mathbb{R}^3$. Let $\theta_t : S^2 \to S^2$ be the flow which is given by a rotation of angle t counterclockwise about the z-axis.

   (a) Show that θ_t preserves the symplectic form on S^2, i.e., $\theta_t^* \omega = \omega$.

   (b) Determine the smooth vector field X on S^2 which generates θ_t.

   (c) Find a Hamiltonian function H of the flow, i.e., a smooth function H such that $dH = i_X \omega$.

7. Use the Mayer-Vietoris sequence and induction to compute the de Rham cohomology groups of the complex projective spaces CP^n, $n \geq 1$. (You may use the fact that $H^p_{DR}(S^n) = 0$ if $0 < p < n$ and $H^p_{DR}(S^n) = \mathbb{R}$ if $p = 0$ or $p = n$.)

8. Consider the vector space $\mathbb{R}^3$ with the cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.

   (a) Show that $(\mathbb{R}^3, \times)$ is a Lie algebra.

   (b) Give an example of Lie group G whose Lie algebra is isomorphic to $(\mathbb{R}^3, \times)$ and write down an explicit isomorphism between the two Lie algebras.