NAME:

Advanced Analysis Qualifying Examination
Department of Mathematics and Statistics
University of Massachusetts

Friday, January 17, 2014

Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question and on the blank page after each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.

2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. (a) Let f be a bounded continuous function mapping $[0, \infty)$ into $\mathbb{R}$. Prove that

$$\lim_{n \to \infty} \int_0^\infty ne^{-nx} f(x)\, dx = f(0).$$

Justify your calculation.

2. Let $X = \{1,2,3,4\}$ and let $\mathcal{M}$ be the smallest σ-algebra of subsets of X that contains the sets $\{1\}$ and $\{1,2\}$.

   (a) List the sets in $\mathcal{M}$.

   (b) Give an example of a function $g : X \to \mathbb{R}$ that is not measurable with respect to $\mathcal{M}$.

   (c) Define the function $f : X \to \mathbb{R}$ by $f(x) = (x-3)(x-4)$, and let μ be a measure on $(X, \mathcal{M})$ with $\mu(\{1\}) = 3$ and $\mu(\{1,2\}) = 8$. Calculate the integral $\int_X f \, d\mu$.

3. All the spaces $L^p[0,1]$ in this problem are defined with respect to Lebesgue measure on $[0,1]$.

(a) Prove that $L^4[0,1] \subset L^3[0,1]$.

(b) Assume that $\Lambda : L^3[0,1] \to \mathbb{R}$ is a bounded linear functional. Prove that the restriction of Λ to $L^4[0,1]$ is a bounded linear functional on $L^4[0,1]$.

(c) Give an example (with proof) of a function in $L^{4/3}[0,1]$ that is not in $L^2[0,1]$.

(d) Give an example (with proof) of a bounded linear functional on $L^3[0,1]$ that is not the restriction to $L^3[0,1]$ of a bounded linear functional on $L^2[0,1]$.

Hint. For parts (b) and (d) use the theorem that characterizes the dual space of $L^p[0,1]$ for $1 < p < \infty$.

4. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let ν be a measure on $\mathcal{M}$ satisfying $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$.

   (a) Prove that ν is absolutely continuous with respect to μ.

   (b) Let $f = \frac{d\nu}{d\mu}$ be the Radon-Nikodym derivative of ν with respect to μ. Prove that $f(x) \leq 1$ a.e. on X with respect to μ.

5. Let $C([0,1])$ denote the set of continuous functions mapping $[0,1]$ into $\mathbb{R}$. Consider the normed vector spaces $(C([0,1]), \|\cdot\|_1)$ and $(C([0,1]), \|\cdot\|_2)$, where for $f \in C([0,1])$

$$\|f\|_1 = \int_0^1 |f(x)|\, dx \text{ and } \|f\|_2 = \sup\{|f(x)| : x \in [0,1]\}.$$

Also let $I : C([0,1]) \to C([0,1])$ denote the identity mapping $I(f) = f$.

(a) Is the mapping $I : (C([0,1]), \|\cdot\|_1) \to (C([0,1]), \|\cdot\|_2)$ continuous? Justify your answer either by a proof (if the assertion is true) or by a counterexample (if the assertion is false).

(b) Is the mapping $I : (C([0,1]), \|\cdot\|_2) \to (C([0,1]), \|\cdot\|_1)$ continuous? Justify your answer either by a proof (if the assertion is true) or by a counterexample (if the assertion is false).

(c) One of the two normed vector spaces $(C([0,1]), \|\cdot\|_1)$ and $(C([0,1]), \|\cdot\|_2)$ is complete and the other is not. Which is which? Explain your answer, but you don't have to provide any detailed proofs.

6. Consider the real Hilbert space $L^2[-1,1]$ defined with respect to Lebesgue measure on $[-1,1]$.

(a) Determine an orthonormal set $\{\varphi_0, \varphi_1, \varphi_2)$ in $L^2[-1,1]$ such that the linear span of $\{1, x, x^2\}$ coincides with the linear span of $\{\varphi_0, \varphi_1, \varphi_2)$.

(b) Compute

$$\min_{a,b,c\in\mathbb{R}} \int_{-1}^{1} \left|x^3 - a - bx - cx^2\right|^2 \, dx.$$

(c) Compute

$$\max \int_{-1}^{1} x^3 \, g(x) \, dx,$$

where $g \in L^2[-1,1]$ is subject to the restrictions

$$\int_{-1}^{1} g(x) \, dx = 0, \; \int_{-1}^{1} x \, g(x) \, dx = 0, \; \int_{-1}^{1} x^2 \, g(x) \, dx = 0, \;\text{ and }\; \int_{-1}^{1} |g(x)|^2 \, dx = 1.$$

7. Let $\mathcal{P}_{\mathbb{N}}$ denote the set of measures θ on the set of positive integers $\mathbb{N}$ satisfying $\theta(\mathbb{N}) = 1$. Any $\theta \in \mathcal{P}_{\mathbb{N}}$ has the form $\theta = \sum_{j\in\mathbb{N}} \theta_j \delta_j$, where $\theta_j = \theta(\{j\})$ and $\delta_j(\{k\}) = 1$ if $k = j$ and $\delta_j(\{k\}) = 0$ if $k \neq j$. The triplet $(\mathbb{N}, \mathcal{M}(\mathbb{N}), \mu)$ is a measure space, where $\mathcal{M}(\mathbb{N})$ is the σ-algebra of all subsets of $\mathbb{N}$.

   (a) Take $\theta \in \mathcal{P}_{\mathbb{N}}$ and let $f : \mathbb{N} \to \mathbb{R}$ be a bounded function. Prove that f is integrable with respect to θ and that
$$\int_{\mathbb{N}} f \, d\theta = \sum_{j=1}^{\infty} f(j)\, \theta_j.$$

   The following definition applies to parts (b) and (c). Let $\{\theta^{(n)}, n \in \mathbb{N}\}$ be a sequence in $\mathcal{P}_{\mathbb{N}}$ and let θ be in $\mathcal{P}_{\mathbb{N}}$. We say that $\theta^{(n)}$ converges weakly to θ, and write $\theta^{(n)} \Rightarrow \theta$, if for all bounded functions $f : \mathbb{N} \to \mathbb{R}$
$$\lim_{n\to\infty} \int_{\mathbb{N}} f \, d\theta^{(n)} = \int_{\mathbb{N}} f \, d\theta.$$

   (b) Prove that if $\theta^{(n)} \Rightarrow \theta$, then for each $j \in \mathbb{N}$, $\lim_{n\to\infty} \theta_j^{(n)} = \theta_j$.

   (c) Fix a number $c \in (1, \infty)$. Let $\{\theta^{(n)}, n \in \mathbb{N}\}$ be a sequence in $\mathcal{P}_{\mathbb{N}}$ satisfying $\int_{\mathbb{N}} x \, d\theta^{(n)} = c$ for all $n \in \mathbb{N}$. Let θ be in $\mathcal{P}_{\mathbb{N}}$. Prove that if $\theta^{(n)} \Rightarrow \theta$, then
$$c = \liminf_{n\to\infty} \int_{\mathbb{N}} x \, d\theta^{(n)} \geq \int_{\mathbb{N}} x \, d\theta.$$

   (d) Give an example of a sequence $\{\theta^{(n)}, n \in \mathbb{N}\}$ in $\mathcal{P}_{\mathbb{N}}$ and a measure $\theta \in \mathcal{P}_{\mathbb{N}}$ with the following three properties: (i) $\theta^{(n)} \Rightarrow \theta$, (ii) $c = \int_{\mathbb{N}} x \, d\theta^{(n)}$ for all $n \in \mathbb{N}$, and (iii) $c > \int_{\mathbb{N}} x \, d\theta$ (i.e., strict inequality in the displayed equation in part (c)).

8. Let $(X, \mathcal{M})$ be a measurable space and let $\{\mu_n, n \in \mathbb{N}\}$ be a sequence of uniformly bounded measures on $\mathcal{M}$; i.e., there exists a constant $b \in (0, \infty)$ such that $\mu(X) \leq b$ for all $n \in \mathbb{N}$. Prove that

$$\mu = \sum_{n=1}^{\infty} \frac{\mu_n}{2^n}$$

is a measure on $\mathcal{M}$ and that each μ_n is absolutely continuous with respect to μ.