DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
January 2013

Do five of the following problems. All problems carry equal weight. Please indicate on the front of your booklet which problems are to be graded. Passing level:

Masters: 60% with at least two substantially correcct

PhD: 75% with at least three substantially correct.

1. Consider Newton's method applied to $f(x) = x^3 - x^2$, which has two distinct roots. Find the largest interval around each root that is part of the basin of attraction for that root. What is the rate of convergence of Newton's method at each root?

2. The harmonic oscillator
$$\dot{q} = p \qquad \dot{p} = -\omega^2 q, \qquad \omega > 0,$$
has bounded solutions for all time. We say an integrator
$$\begin{bmatrix} q^{n+1} \\ p^{n+1} \end{bmatrix} = \psi_h \left(\begin{bmatrix} q^n \\ p^n \end{bmatrix} \right)$$
is asymptotically stable if q^n and p^n are bounded for all time. For both the explicit Euler and implicit midpoint integrators, find all values $h > 0$ such that the integrator applied to the harmonic oscillator is asymptotically stable.

3. Consider the function $f(x) = e^x$ over the interval $[0, 1]$.

   (a) Find the Hermite interpolation polynomial $H(x)$ that satisfies: $H(0) = f(0)$, $H'(0) = f'(0)$, $H(1) = f(1)$, $H'(1) = f'(1)$.

   (b) Derive a good upper bound for the infinity norm error, i.e.,
$$\max_{0 \le x \le 1} |e^x - H(x)|.$$

4. Consider
$$u_{xx} + u_{yy} = f$$
with Dirichlet boundary condition on a rectangular domain. Write down the linear system when the second order central difference is used to discretize both derivatives. Is the Gauss-Seidel iteration convergent for the above system? Prove your answer. ($\Delta x = \Delta y$ can be assumed if that makes the proof easier.)

5. An $n \times n$ complex matrix A is said to be *normal* if it commutes with its conjugate transpose, i.e. $AA^* = A^*A$, where $A^* = \bar{A}^T$.

   (a) Schur's Theorem states that any complex matrix is unitarily similar to an upper triangular matrix. Use Schur's theorem to prove the following *Spectral Theorem for Normal Matrices* for a given matrix A:

   $$A \text{ is normal} \iff A \text{ is unitarily similar to a diagonal matrix}$$

   (b) Given *any* square matrix A it can be shown that

   $$\lim_{k \to \infty} A^k = 0 \iff \rho(A) < 1,$$

   where $\rho(A)$ is the *spectral radius* of A. Prove this result in the case where A is assumed to be normal.

6. Find the coefficients $\{a_0, a_1, a_2, a_3\}$ and nodes $\{x_1, x_2\}$ so that the quadrature formula

   $$\int_{-1}^{1} f(x)dx \approx a_0 f(-1) + a_1 f(x_1) + a_2 f(x_2) + a_3 f(1)$$

   has the highest possible degree of precision, i.e., is exact for $\{1, x, x^2, \ldots, x^p\}$, where p is the largest value possible.

7. Let $u, v \in \mathbb{R}^n$ be column vectors, and consider the *rank one perturbation of the identity* defined by $A = I - uv^T$.

   (a) Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^T$ for some scalar α. Give an expression for α.

   (b) For what u and v is A singular? Show if A is singular, then it is a projector.

   (c) For what u and v is A an orthogonal projector?