UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
ADVANCED EXAM - Mathematical Statistics and Probability
January 16, 2013

70 points are required to pass with at least 25 points from each part; Part 1 = probability (questions 1-3) and Part 2 = Multivariate/Linear Models (questions 3 and 4).

**Probability Theory**

1. (a) (20 PTS). Prove the first Borel-Cantelli Lemma: If $\{A_n\}$ is a sequence of events in a probability space $(\Omega, P)$ with $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup A_n) = P(A_n i.o) = 0$.

   (b) Use the above lemma to prove the Strong law of large numbers: If $\{X_n\}$ are i.i.d with $E(X_i = 0)$ and $E(X_i^4) < \infty$ and $S_n = X_1 + \cdots + X_n$, then $S_n/n \to 0$ a.s. (almost surely).

2. (15 PTS).

   (a) Suppose $X_1, \ldots, X_n$ are independent positive random variables on $\Omega$, with finite expectations. Prove $E(\Pi_{i=1}^n X_i) = \Pi_{i=1}^n E(X_i)$.

   (b) Provide an example of two dependent random variables $X$ and $Y$, such that $E(XY) = E(X)E(Y)$.

3. (15 PTS). Let $S_n$ be an i.i.d. sequence $\{X_n\}$ with mean $\mu$ and variance $\sigma^2$. Let $N$ be a random variable that is independent of all the $X_n$ and takes on the values $1, 2, 3, \cdots$. Let $S_N = \sum_{i=1}^N X_i$. Note that the number of terms is random.

   a) Find $E[S_N|N]$ and $E[S_N^2|N]$.

   b) Now find $E(S_N)$ and $E(S_N^2)$ with your answers given in terms of $\mu$, $\sigma^2$ and the mean and variance of $N$.

   In each case explain how you arrive at your answer.

**Multivariate distribution theory/Linear Models**

4. (30 PTS) Consider $\mathbf{Y}$ $(n \times 1)$ distributed multivariate normal with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

   (a) State what the moment generating function of $\mathbf{Y}$ is.

   (b) Using moment generating functions, **derive** the distribution of $\boldsymbol{a}'\mathbf{Y}$, where $\boldsymbol{a}$ is a $n \times 1$ vector of constants.

   (c) Derive $E(\mathbf{Y}'\mathbf{A}\mathbf{Y})$ where $\mathbf{A}$ is a symmetric matrix. Your answer should involve $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ and $\mathbf{A}$.

   (d) Now suppose

   $$\begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right).$$

   Define $\mathbf{W} = (\mathbf{Y}_1 - \boldsymbol{\mu}_1) - \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{Y}_2 - \boldsymbol{\mu}_2)$.

   i. Show $\mathbf{W} \sim N(\mathbf{0}, \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$.

   ii. Show $Cov(\mathbf{W}, \mathbf{Y}_2) = \mathbf{0}$.

   iii. Give the conditional distribution of $\mathbf{W}$ given $\mathbf{Y}_2 = \mathbf{c}$ *but WITHOUT doing it by writing out the joint distribution of* $\mathbf{W}$ *and* $\mathbf{Y}_2$ *and using conditional distributions for the multivariate normal.* Hint: Make use of the previous result about $\mathbf{Y}_2$ and $\mathbf{W}$? Explain your answer.

5. (20 PTS) Let $\mathbf{Z}$ $(n \times 1)$ have a standard multivariate normal distribution. Let $\mathbf{A}$ $(n \times n)$ be symmetric and idempotent with rank $p \leq n$.

   (a) Prove that $\mathbf{Z}'\mathbf{A}\mathbf{Z}$ has the same distribution as $U = \mathbf{Z}_1'\mathbf{Z}_1$ where $\mathbf{Z}_1$ $(p \times 1)$ has a standard multivariate normal distribution.

   (b) Show that the moment generating function of $U$ is $m_U(t) = (1 - 2t)^{-p/2}, 1 - 2t > 0$. What is $U$'s distribution?

   (c) Find a non-trivial $k \times n$ matrix $\mathbf{B}$ (i.e., $\mathbf{B}$ not $\mathbf{0}$) so that $\mathbf{B}\mathbf{Z}$ and $\mathbf{Z}'\mathbf{A}\mathbf{Z}$ are independent. (For full credit, you are asked to find a $\mathbf{B}$ and prove independence.)