University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
For January 2013

Do 5 of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Prove or disprove the following statements:

   (a) The tangent bundle $T(S^2 \times S^1)$ is trivial.

   (b) If a 1-form α on a smooth manifold M is nowhere zero and θ is another 1-form such that $\theta \wedge \alpha = 0$, then there exists $f \in C^\infty(M)$ such that
   $$\theta = f\,\alpha.$$

   (c) If a 1-form α on $M = \mathbb{R}^2 \backslash \{0\}$ such that $d\alpha = 0$, then there exists $f \in C^\infty(\mathbb{R}^2 \backslash \{0\})$ such that $\alpha = df$.

2. Let $M_n(\mathbb{R})$ be the space of real $n \times n$ matrices. Show that the identity matrix I is a regular value of the smooth map $M_n(\mathbb{R}) \to S_n(\mathbb{R}) : A \mapsto AA^t$, where $S_n(\mathbb{R})$ is the space of symmetric $n \times n$ matrices. Use this to prove $O_n(\mathbb{R}) = \{A : AA^t = I\} \subset M_n(\mathbb{R})$ is a smooth submanifold of dimension $n(n-1)/2$.

3. Let X_1, X_2, X_3 be smooth vector fields defined in a neighborhood U of $0 \in \mathbb{R}^3$. Suppose

   (a) $X_1(0), X_2(0), X_3(0)$ are linearly independent;

   (b) $[X_i, X_j] \equiv 0$ for all i, j.

   Show that there are local coordinate functions x_1, x_2, x_3 defined in a neighborhood $V \subset U$ of $0 \in \mathbb{R}^3$ such that
   $$X_i = \frac{\partial}{\partial x_i}, \text{ for } i = 1, 2, 3.$$

4. Consider the vector space $\mathbb{R}^3$ with the cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.

   (a) Show that $(\mathbb{R}^3, \times)$ is a Lie algebra.

   (b) Determine the Lie group G such that (i) G is simply connected, (ii) the Lie algebra of G is isomorphic to $(\mathbb{R}^3, \times)$.

5. Use the Mayer-Vietoris sequence and induction to compute the de Rham cohomology groups of the n-sphere S^n and the real projective space $\mathbb{R}P^n$.

6. Let M be a closed $2n$-dimensional smooth manifold and let ω be a closed 2-form on M which is non-degenerate, i.e., for any $p \in M$, $X \in T_pM$, $X \mapsto i_X\omega(p)$ defines an isomorphism between T_pM and T_p^*M. Show that $H^{2k}_{dR}(M) \neq 0$ for any $0 \leq k \leq n$.

7. Prove that any smooth, proper function $f : \mathbb{C} \setminus \{0, 1\} \to \mathbb{R}$ has a critical point (recall that f is proper if the pre-image of any compact set is compact).

8. Endow the upper half-plane

$$H = \{(x, y) : y > 0\}$$

with the Riemannian metric $(dx^2 + dy^2)/y^2$.

   (a) Show that vertical rays

$$\{(x, y) : x = a\} \subset H$$

   and half-circles

$$\{(x, y) : (x - b)^2 + y^2 = c\} \subset H$$

   are geodesics.

   (b) Compute the Gauss curvature of H.

   (c) Apply the Gauss-Bonnet formula to find the area of the noncompact domain

$$\{(x, y) : -1 < x < 1, x^2 + y^2 > 1\} \subset H.$$