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ADVANCED EXAM — DIFFERENTIAL EQUATIONS
JANUARY 2013

Do five of the following seven problems. All problems carry equal weight. Passing level: 75% with at least three substantially complete solutions.

1. Consider the system

$$\dot{x} = y\,z, \quad \dot{y} = -z\,x, \quad \dot{z} = -z^3,$$

in $\mathbb{R}^3$, and let $v = (x_0, y_0, z_0) \in \mathbb{R}^3$ be arbitrary.

   (a) Solve the equations by first transforming to cylindrical coordinates.

   (b) Determine the forward orbit of v and use this to describe the ω-limit set of (the orbit of) v.

   (c) If the equation for z is replaced by $\dot{z} = -z$, find the new orbit and ω-limit set of v.

2. The Lorenz equations are

$$\dot{x} = \sigma\,(y - x), \quad \dot{y} = r\,x - y - x\,z, \quad \dot{z} = x\,y - b\,z,$$

where r, σ and b are positive constants.

   (a) For $r < 1$, show that the origin is globally asymptotically stable by considering the function $V_1(x, y, z) = \alpha\,x^2 + \beta\,y^2 + \gamma\,z^2$.

   (b) By considering the function $V_2(x, y, z) = r\,x^2 + \sigma\,y^2 + \sigma\,(z - 2\,r)^2$, show for general r that all trajectories eventually enter and then remain within a bounded ellipsoid in phase space.

3. Calculate the stable and unstable manifolds of the origin to third order (i.e., up to and including the cubic terms) for the system

$$\dot{x} = -x + y^2, \quad \dot{y} = y - x^2.$$

Sketch the phase portrait for $|x| \ll 1$, showing the curvature of the two manifolds. Find and classify the other fixed point, and sketch the phase portrait on the scale $|x| = O(1)$. You may find it useful to also sketch the null-clines.

4. Consider the initial-boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= x \qquad (\, 0 < x < 1\,, \quad t > 0\,) \\ u(0,t) = u(1,t) &= 0, \qquad u(x,0) = 0. \end{aligned} \tag{1}$$

Exhibit the solution to the IBVP (1) using a Fourier series method.

5. **(a)** Let $u(x,t)$ by any smooth solution of the following PDE (which is a wave equation with two damping terms):

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial^3 u}{\partial x^2 \partial t} = 0\,, \tag{2}$$

where α and β are positive constants. Assume that the solution exists in all of space ($-\infty < x < +\infty$) for all $t \geq 0$, and that it vanishes rapidly as $|x|$ goes to infinity. Show that the energy of the solution,

$$E(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \left[\frac{\partial u}{\partial t}\right]^2 + \left[\frac{\partial u}{\partial x}\right]^2 dx\,,$$

decreases in $t > 0$.

(b) Use part (a) to prove the uniqueness of smooth solution of the initial value problem for (2) (assuming the rapid decay of all solutions as $|x|$ goes to infinity).

6. Let $u(x,y)$ be a harmonic function in a bounded planar domain $\Omega \subset R^2$ with smooth boundary $\partial\Omega$. Assume that $u \in C^3(\Omega \cup \partial\Omega)$.

(a) Consider $v = |\nabla u|^2$, and show that v satisfies

$$\Delta v \geq 0 \quad \text{ in } \Omega\,.$$

(b) Use part (a) to establish the gradient estimate

$$\sup_{\Omega} |\nabla u| \leq \sup_{\partial\Omega} |\nabla u|\,.$$

7. **(a)** Describe the weak formulation of the Neumann boundary value problem

$$\begin{aligned} -\Delta u &= f(x) \quad \text{in } \Omega\,, \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{3}$$

for a smoothly bounded domain $\Omega \subset R^n$, with outward unit normal $\mathbf{n}$ on $\partial\Omega$. Introduce the appropriate Sobolev space for weak solutions u, and explain how the weak form of the BVP is derived from the classical PDE and its boundary conditions.

(b) Prove that the BVP (3) has a weak solution for data $f \in L^2(\Omega)$ if and only if $\int_\Omega f\,dx = 0$.

(c) To what extent are such solutions unique?