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Advanced Analysis Qualifying Examination
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Wednesday, January 16, 2013

Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.

2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let $\mathcal{B}$ be the Borel σ-algebra of subsets of $\mathbb{R}$ and let m be Lebesgue measure.

   (a) For $A \in \mathcal{B}$ and $c \in \mathbb{R}$ let $A + c = \{x \in \mathbb{R} \,:\, x = a + c \text{ for some } a \in A\}$. Prove that $A + c \in \mathcal{B}$.
   Hint: Consider the set $\mathcal{D} = \{A \in \mathcal{B} \,:\, A + c \in \mathcal{B} \text{ for all } c \in \mathbb{R}\}$.

   (b) Prove that for any $A \in \mathcal{B}$ and $c \in \mathbb{R}$ we have

   $$m(A + c) = m(A)\,.$$

   (c) Let f be an integrable function. Prove that for any $A \in \mathcal{B}$

   $$\int_A f(x + c)dm(x) \;=\; \int_{A+c} f(x)dm(x)\,.$$

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to [0, \infty)$ a non-negative measurable function. For $k \in \mathbb{Z}$ consider the level sets

$$F_k \,=\, \{x \in X \,:\, 2^k < f(x) \leq 2^{k+1}\}\,, \qquad E_k \,=\, \{x \in X \,:\, 2^k < f(x)\}\,.$$

Show that the following are equivalent.

(a) $f \in L^1(\mu)$.

(b) $\displaystyle\sum_{k=-\infty}^{\infty} 2^k \mu(F_k) \,<\, \infty.$

(c) $\displaystyle\sum_{k=-\infty}^{\infty} 2^k \mu(E_k) \,<\, \infty.$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ sequences of measurable functions, and f and g measurable functions. Assume that $f_n \to f$ in measure and $g_n \to g$ in measure.

   (a) Prove that for any $a, b \in \mathbb{R}$, $af_n + bg_n \to af + bg$ in measure.

   (b) Prove that $|f_n| \to |f|$ in measure.

   (c) Assume that there exists $M < \infty$ such that $|f_n| \leq M$ a.e, $|g_n| < M$ a.e. for all n and $|f| \leq M$ a.e, $|g| \leq M$ a.e. Prove that $f_n g_n \to fg$ in measure.

4. Let $[a, b]$ be a finite interval and $f : [a, b] \to \mathbb{R}$.

(a) Give the definition of the concept that "f is of bounded variation."

(b) Suppose that f is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and that $\sup_{x\in(a,b)} |f'(x)| < \infty$. Prove that f is of bounded variation.

(c) Prove that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if} \quad 0 < x \leq 1/\pi \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is of bounded variation.

(d) Prove that the function

$$f(x) = \begin{cases} x \sin(1/x) & \text{if} \quad 0 < x \leq 1/\pi \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is not of bounded variation.

5. Consider the measurable space $(\mathbb{N}, \mathcal{P})$, where $\mathcal{P}$ is the σ-algebra of all subsets of $\mathbb{N}$. Let μ and λ two measures on $(\mathbb{N}, \mathcal{P})$.

   (a) Find, and describe explicitly, two subsets of $\mathbb{N}$, I_1 and I_2, such that the following three properties hold:

      i. The measure λ_1 defined by $\lambda_1(E) \equiv \lambda(E \cap I_1)$ for any $E \in \mathcal{P}$ satisfies $\lambda_1 \ll \mu$.

      ii. The measure λ_2 defined by $\lambda_2(E) \equiv \lambda(E \cap I_2)$ for any $E \in \mathcal{P}$ satisfies $\lambda_2 \perp \mu$.

      iii. $\lambda = \lambda_1 + \lambda_2$.

   (b) Find an explicit formula for a function $h : \mathbb{N} \to \mathbb{R}$ such that, for all $E \in \mathcal{P}$,

$$\lambda_1(E) = \int_E h d\mu \,.$$

   Is h unique?

6. Let $1 \leq p < q < r \leq \infty$ and let $L^p = L^p(X, \mathcal{M}, \mu)$ for some measure space $(X, \mathcal{M}, \mu)$. L^q and L^r are defined similarly.

   (a) Show that $L^q \subset L^p + L^r$; i.e., any $f \in L^q$ can be written as $f = g + h$ with $g \in L^p$ and $h \in L^r$.

   (b) Show that $L^p \cap L^r \subset L^q$ and that for any $f \in L^p \cap L^r$, $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$ for a suitable λ.
   Hint: Use Hölder's Inequality.

7. Let $(X, \|\cdot\|)$ be a Banach space. Suppose that X can be written as the direct sum of two linear subspaces M and N; i.e. $X = M \oplus N$. This means that $x \in X$ can be expressed uniquely as $x = y + z$ where $y \in M$ and $z \in N$.

   (a) For any $x \in X$ define $\|x\|' \equiv \|y\| + \|z\|$. Prove that $\|\cdot\|'$ defines a norm on X.

   (b) Consider the normed vector space $(X, \|\cdot\|')$. Which additional property of the subspaces M and N is needed to ensure that $(X, \|\cdot\|')$ is a Banach space? Prove your answer.

8. (a) Suppose that $f \in L^1([0,1])$, and define for $n \in \mathbb{Z}$

$$c_n = \int_0^1 f(x) e^{-i2\pi nx} dm(x) .$$

Prove the Riemann-Lebesgue lemma; i.e., that

$$\lim_{n\to\infty} c_n = 0 .$$

Hint: Approximate f by a function in L^2 and use Hilbert space theorems.

(b) Suppose that E is a measurable subset of $[0,1]$ and $\{u_n\}_{n\in\mathbb{N}}$ an arbitrary sequence of real numbers. Prove that

$$\lim_{n\to\infty} \int_E \cos^2(nx + u_n)\, dm(x) = \frac{m(E)}{2} .$$

Hint: $\cos^2 a = \frac{1}{2}(1 + \cos(2a)),\ \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.