Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

January 18, 2012

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

Throughout this exam, $\mathbb{R}$ denotes the real line with the standard topology.

(1) (a) Define the *uniform topology* on a product $\prod_{i\in I} X_i$ of metric spaces.
(b) Let A be a countable set and let $Z = \prod_{\alpha\in A} I_\alpha$, where $I_\alpha = [0,1] \subset \mathbb{R}$. Give Z the uniform topology. Find an infinite subset of Z that has no limit point.

(2) Prove that no two of the following three spaces are homeomorphic: (i) $\mathbb{R}$, (ii) $\mathbb{R}^2$ (both with the standard topology), and (iii) $\mathbb{R}^\omega$, the product of countably many copies of $\mathbb{R}$, with the product topology.

(3) (a) Define what it means for a sequence of functions to *converge uniformly*.
(b) Let X be a topological space and Y be a metric space. Let $f_n \colon X \to Y$ be a sequence of functions. Prove that if f_n converges uniformly to f, then f is continuous.

(4) A subspace $X \subset \mathbb{R}^n$ is *convex* if for all $x, y \in X$ and all $t \in [0,1]$, we have $tx + (1-t)y \in X$. Show that any convex subspace is simply-connected.

(5) Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ denote the 2-sphere. Let $p, q \in S^2$ be two distinct points. Let X denote the quotient space after identifying the two points. Compute $\pi_1(X)$.

(6) Let (X,d) be a metric space, and let $f\colon X \to X$ be a continuous function without any fixed points.
(a) If X is compact, show that there exists an $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.
(b) Give an example to show that this is not true if X is not compact.

(7) (a) Prove that a compact locally-connected space has only finitely many components.
(b) Show with two examples that both conditions are necessary.