DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERICAL ANALYSIS EXAM
JANUARY 2012

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correct
PhD: 75% with at least three substantially correct

1. Determine the values of α for which the matrix

$$A = \begin{pmatrix} 9 & 6 & 9 \\ 6 & 20 & 26 \\ 9 & 26 & \alpha \end{pmatrix}$$

is positive definite.

2. Consider using Newton's method to find the root of the polynomial

$$p(x) = x^2 - 2x + 1$$

   (a) Does the Newton iteration converge for all initial guesses? Justify your answer.
   (b) When it converges what is the rate of convergence? Justify your answer.

3. $f(x)$ is a polynomial of degree at most 3 whose value at 9 distinct points is given below:

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-9	0	2	0	-3	-4	0	12	35

Find the exact value of $\int_{-4}^{4} f(x)\,dx$. Explain how you are sure that your answer is correct.

4. State the Gauss quadrature for the following integral

$$\int_0^\infty f(x)\,e^{-x}\,dx,$$

and derive the one with exactly two nodes.

5. Consider the following 2×2 matrix

$$A = \begin{pmatrix} a & -b \\ -a & a \end{pmatrix}$$

where a and b are real numbers satisfying $a > b > 0$. Show that the Gauss-Seidel iteration method is convergent for this class of matrices.

6. Let $x_0 < x_1 < \ldots < x_n$ be $n+1$ distinct points.

   (a) Given a function $f(x)$ prove that there is a unique polynomial of degree at most n that interpolates $f(x)$ at the node points x_i.

   (b) Derive the Lagrange form of the interpolation polynomial.

   (c) Find the lowest order polynomial $p(x)$ that satisfies the conditions:

$$p(0) = 3, \qquad p(1) = 4, \qquad p'(0) = -1, \qquad p'(1) = 3.$$

7. Consider the ODE initial-value problem

$$\frac{dy}{dx}(x) = f(x, y(x)),$$

with initial data $y(x_0) = y_0$. We would like to solve this initial value problem at points $x_n = nh, n = 0, ..., N$ where $h = x_n - x_{n-1}$ for all n. Find the highest order method in the class

$$y_{n+1} = y_n + h[b_1 f(x_n, y_n) + b_2 f(x_{n-1}, y_{n-1})].$$

i.e., find b_1 and b_2 for the above method which gives the highest order local truncation error. State the order of the method obtained.