UMASS AMHERST – BASIC EXAM – COMPLEX ANALYSIS

JANUARY 20, 2012

Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded. **Notation**: We denote by $\mathbb{D}_r(z_0)$ the open disc of radius r centered at z_0, i.e. $\mathbb{D}_r(z_0) = \{z \in \mathbb{C} \ : \ |z - z_0| < r\}$. We let $\mathbb{D} = \mathbb{D}_1(0)$ be the unit disc; the path γ is its boundary, traversed once counterclockwise.

---

(1) Prove that the following statement is false: For every function f holomorphic on the annulus $1/2 < |z| < 3/2$, there is a polynomial p such that $|f - p| < 1/2$ on the unit circle, i.e. $|f(z) - p(z)| < 1/2$ for $|z| = 1$.
Hint: Let $f(z) = 1/z$.

---

(2) Consider the strip $U = \{z \in \mathbb{C} \mid -1 < \mathrm{Re}(z) < 1\}$.
  (a) Find an explicit formula for a one-to-one conformal map f of U onto the unit disc $\mathbb{D}$ for which $f(0) = 0$ and $f'(0)$ is a positive real number.
  (b) Compute $f'(0)$ for the function f you provided for part (a).
  (c) Is the function f of part (a) uniquely determined?

---

(3) Use contour integration to prove that for $n \geq 1$

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^{2n}} = \frac{\pi}{n} \csc(\frac{\pi}{2n}).$$

Hint: You may wish to use a "pizza slice"-shaped wedge with angle π/n.

---

(4) Prove that if f is a holomorphic map $f : \mathbb{D}_R(0) \to \mathbb{D}_M(0)$, for some positive real numbers R and M, then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)} f(z)} \right| \leq \frac{|z|}{MR}.$$

---

(5) Suppose f is holomorphic on $\mathbb{D}$ and satisfies $|f'(z) - f'(0)| < |f'(0)|$ for all $z \in \mathbb{D}$. Prove that f is one-to-one on $\mathbb{D}$.

---

(6) Let us write the Taylor expansion of the function $f(z) = z/(e^z - 1)$ at $z = 0$ as follows:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} b_n z^n.$$

(a) Determine the radius of convergence of this series.
(b) Using the Cauchy Integral Formula and the contour $\gamma : |z| = 1$, for every $n \geq 0$, find an integral expression of the form

$$b_n = \int_0^{2\pi} g_n(\theta) d\theta$$

for the complex numbers b_n, where $g_n : [0, 2\pi] \to \mathbb{C}$ is a suitable function. Give the function g_n explicitly.[1]

---

(7) Suppose f is an entire function such that $f(0) = 1$.
(a) Prove that if there is a constant a such that $f'(z) = af(z)$ for all z, then $f(z) = e^{az}$.
(b) Prove that if $f(z_1 + z_2) = f(z_1)f(z_2)$ for all $z_1, z_2 \in \mathbb{C}$, then there is a constant a such that $f(z) = e^{az}$.

---

(8) (a) State and prove the Casorati-Weierstrass theorem about the behavior of a function in a neighborhood of an essential singularity.
(b) Let $f(z)$ be a non-constant entire function. Classify all the singularities of $g(z) := e^{f(1/z)}$ in the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

---

(9) Let f be a non-constant holomorphic function on a region R which contains the closed unit disk $\overline{\mathbb{D}}$. Assume that $|f| = 1$ on the unit circle i.e. $|f(z)| = 1$ if $|z| = 1$. Prove that the image $f(\overline{\mathbb{D}})$ of $\overline{\mathbb{D}}$ under f contains the unit disk $\mathbb{D}$.

---

(10) (a) Find the Taylor series expansion of the function $f(z) = \sin^2(z)$ around the origin. For what values of z does the series converge?
(b) Find the Laurent series expansion of the function $f(z) = \dfrac{1}{z^2 - 3z + 2}$ valid near (and centered at) the point $z_0 = 1$. For what values of z does the series converge?

---

[1] The numbers $n!b_n$ are called the *Bernoulli numbers* and play a special role in number theory.