University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
January 2012

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Let $F\colon \mathbb{P}^2 \to \mathbb{P}^4$ be the map

$$F([x : y : z]) = [x^2 : xy : xz + y^2 : yz : x^2 + y^2 + z^2].$$

   (a) Prove that F is an embedding.

   (b) Show how F may be used to define an embedding $\varphi\colon \mathbb{P}^2 \to \mathbb{R}^4$.

2. Let G be a Lie group.

   (a) Let $\varphi\colon G \to G$ be a Lie group automorphism. Prove that φ_* maps left-invariant vector fields on G to left-invariant vector fields on G.

   (b) Let $\psi\colon G \to G$ be the map $\psi(g) = g^{-1}$. Prove that ψ_* maps left-invariant vector fields on G to right-invariant vector fields on G.

3. Let S^{2n-1} be the unit sphere in $\mathbb{C}^n = \mathbb{R}^{2n}$, i.e. the set of $(z_1, \ldots, z_n)$ such that $\sum |z_i|^2 = 1$. Consider the group $\Gamma = \{1, \omega, \omega^2\}$, where $\omega = e^{2\pi i/3}$. Let M be the quotient of S^{2n-1} by the action of Γ given by

$$\omega(z_1, \ldots, z_n) = (\omega z_1, \ldots, \omega z_n).$$

   (a) Show that M is smooth orientable manifold.

   (b) Show that the homomorphism $H^k_{dR}(M) \to H^k_{dR}(S^{2n-1})$ is injective, with image the Γ-invariants $H^k_{dR}(S^{2n-1})^\Gamma$.

4. Let M and N be manifolds. Prove or disprove the following statements:

   (a) $M \times N$ is orientable if and only if M and N are orientable.

   (b) $M \times N$ is parallelizable if and only if M and N are parallelizable.

5. Let $\pi\colon E \to M$ be a vector bundle and $F \subseteq E$ a subbundle. Prove that there exists a subbundle $F' \subseteq E$ such that $F \oplus F' \cong E$. Here $\cong$ means isomorphism of vector bundles.

6. Let $\alpha = y\,dx + dz$, a 1-form on $\mathbb{R}^3$. Prove or disprove the following statements:

   (a) For any $p \in \mathbb{R}^3$, there exists an immersion $f\colon \mathbb{R} \to \mathbb{R}^3$ with $f(0) = p$ and $f^*\alpha = 0$.

   (b) For any $p \in \mathbb{R}^3$, there exists an immersion $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ with $f(0) = p$ and $f^*\alpha = 0$.

7. Take $k > 0$, and consider the metric

$$g = dr^2 + k^2 r^2 d\theta^2$$

on $M = \mathbb{R}^2 \setminus \{(0,0)\}$ in polar coordinates (recall that although the function θ is not well-defined on all of M, its differential $d\theta$ is).

   (a) Show that (M, g) is flat.

   (b) Write differential equations for parallel transport around the loop $r = 1$, $\theta = t$, $0 \leq t \leq 2\pi$.

8. Let (M, g) be an n-dimensional, compact, oriented manifold without boundary. Let Ω_g denote the volume element of (M, g). Given a vector field X on M, its *divergence*, $\operatorname{div}(X)$, is the C^∞ function on M defined by the identity:

$$L_X(\Omega_g) \;=\; \operatorname{div}(X)\,\Omega_g,$$

where L_X denotes the Lie derivative with respect to X.

   (a) Prove that $\int_M \operatorname{div}(X)\,\Omega_g = 0$.

   (b) Express $\operatorname{div}(X)$ in local coordinates.