Department of Mathematics and Statistics
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ADVANCED EXAM — DIFFERENTIAL EQUATIONS

JANUARY 2012

Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1. Consider the system

$$\dot{x} = -x + \frac{y}{\log\sqrt{x^2+y^2}}, \qquad \dot{y} = -y - \frac{x}{\log\sqrt{x^2+y^2}},$$

   on the unit disk.

   (a) Characterize the behavior of the linearization at the origin.

   (b) Using polar coordinates, show that the origin is a stable focus (i.e. spirals inward) for the nonlinear problem.

   (c) Why are the different answers not a contradiction? [Hint: consider smoothness]

2. Consider the damped Hamiltonian system

$$\dot{x} = y, \qquad \dot{y} = -\gamma\, y - x - x^2, \qquad \gamma > 0.$$

   (a) Find a Liapunov function for the system.

   (b) Explain why each solution converges to one of the fixed points or infinity (i.e. eventually leaves any compact set).

   (c) Compute the linear stability of the fixed points, and roughly sketch the phase plane.

   (d) By considering stable and unstable manifolds for different values of γ, show that for some γ, there is a heteroclinic orbit.

3. Find *all* ω-limit sets of the system

$$\dot{x} = -\epsilon\, y + x\,(x^2+y^2)\,\sin\tfrac{\pi}{x^2+y^2}$$
$$\dot{y} = \epsilon\, x + y\,(x^2+y^2)\,\sin\tfrac{\pi}{x^2+y^2}\ ,$$

and identify which ones are stable. [Hint: use polar coordinates]

4. Let $F:\mathbb{R}\to\mathbb{R}$ be a C^1 function such that $F(0)=0$ and $F'(v)>0$ for all $v\in\mathbb{R}$. Let $I=(0,1)$ and let $u:\bar{I}\times[0,T]$ be a smooth solution to the the mixed initial/boundary value problem

$$\begin{cases} u_{tt}-u_{xx}+F(u_t) \;=\; 0 & \text{on } I\times(0,T] \\ u\equiv 0 & \text{on } \{x=0\}\times[0,T)\cup\{x=1\}\times[0,T) \\ u=g,\ \text{and } \partial_t u = h & \text{on } I\times\{t=0\} \end{cases} \tag{1}$$

where $g,\ h\in C_c^\infty(I)$ (smooth and compactly supported).

Let $E[u] := \frac{1}{2}\int_0^1 |u_t|^2+|u_x|^2\,dx$ be the 'energy' associated to (1) where the integrand is understood to be evaluated at (x,t).

(a) Prove that $0\le E(t)\le E(0)$.

(b) Prove the uniqueness of classical solutions to (1).

5. We say that a function $v\in C^2(\overline{\Omega})$ is *subharmonic* if $-\Delta v\le 0$ in Ω.

(a) Prove that if $v\in C^2(\bar{\Omega})$ is subharmonic then

$$v(x)\le \frac{1}{|B(x,r)|}\int_{B(x,r)} v(y)\,dy \qquad \text{for all } B(x,r)\subset\Omega.$$

(b) Prove that therefore, $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$.

(c) Let $\phi:\mathbb{R}\to\mathbb{R}$ be a smooth and <u>convex</u> function. Assume u is harmonic and let $v(x):=\phi(u(x))$. Prove that v is subharmonic.

6. (a) Determine the type (elliptic, parabolic or hyperbolic) of the equation

$$u_{xx} - 2u_{xy} \sin x - u_{yy} \cos^2 x - u_y \cos x = 0$$

and find the characteristic curves (if any).

(b) Solve by the method of characteristics

$$x^2 u_x + y^2 u_y = u^2, \quad u(x, 2x) = 1.$$

7. (a) Consider the following wave equation in three space dimensions

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in} \quad \mathbb{R}^3 \times \mathbb{R} \\ u(x, 0) = 0 & \text{for all} \quad x \in \mathbb{R}^3 \\ u_t(x, 0) = |x|^2 & \text{for all} \quad x \in \mathbb{R}^3. \end{cases}$$

Use Kirchhoff's formula to find the solution $u(x, t)$ explicitly.

(b) Consider the solution $u(x, t)$ to the homogeneous linear wave equation in 3d with smooth initial data $u(x, 0) = g(x)$ and $u_t(x, 0) = h(x)$ **vanishing outside** a ball $B(0, R)$. Use *finite propagation speed* and *Huygens' principle* to determine the region where $u(x, t)$ **certainly** vanishes.