Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

January 10, 2011

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

Let $\mathbb{R}$ denote the real line with the standard topology.

(1) If A is a subspace of X, the boundary of A is $\text{Bd}(A) = \overline{A} \setminus A$.
- (a) Show that the interior of $\text{Bd}\, A$ is empty for any $A \subset X$.
- (b) Let $f\colon X \to Y$ be continuous. Decide whether each of the statements "for any $A \subset X$, $f(\text{Bd}(A)) = \text{Bd}\, f(A)$" and "for any $B \subset Y$, $f^{-1}(\text{Bd}(B)) = \text{Bd}\, f^{-1}(B)$" is true or false, and give a proof or a counterexample for each one.

(2)
- (a) Give an example, with proof, of a space that is locally-connected but not connected.
- (b) Give an example, with proof, of a space that is path-connected but not locally path-connected.

(3) Let X be the quotient space $S^2/\{N, S\}$, where $N = (0,0,1)$, $S = (0,0,-1)$. Let Y be the quotient space $(S^1 \times S^1)/(S^1 \times \{p\})$ for any point $p \in S^1$. Show that X and Y are homeomorphic.

(4) Let X be the union of the three coordinate axes in $\mathbb{R}^3$. Compute

$$\pi_1(\mathbb{R}^3 \setminus X).$$

(5) Given points $x, y \in \mathbb{R}^2$, let

$$L(x,y) = \{tx + (1-t)y \mid 0 \le t \le 1\}$$

be the line segment joining x and y. For an open set $U \subset \mathbb{R}^2$, show that

$$\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid L(x,y) \subset U\}$$

is open in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. (Hint: compactness is important here.)

(6) Let $X = \prod_{\alpha \in A} X_\alpha$ be a product, where each X_α is a topological space.
- (a) Define the *box topology* and *product topology* on X.
- (b) Give a necessary and sufficient condition for an infinite sequence $\{\mathbf{x}_i\} = \{(x_\alpha)_i\}$ to converge in the product topology.
- (c) Give a necessary and sufficient condition for an infinite sequence $\{\mathbf{x}_i\} = \{(x_\alpha)_i\}$ to converge in the box topology.

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(7) Let X, Y be topological spaces, and suppose Y is a metric space with metric d. Let $\mathcal{F}(X, Y)$ be the set of all functions from X to Y. Let $\mathcal{B}(X, Y) \subset \mathcal{F}(X, Y)$ be the space of bounded functions from X to Y. Thus $\mathcal{B}(X, Y)$ consists of all functions $f \in \mathcal{F}(X, Y)$ such that $f(X)$ is a bounded subset of Y. Define δ on $\mathcal{B}(X, Y)$ by

$$\delta(f, g) = \sup_{x \in X}\{d(f(x), g(x))\}.$$

(a) Show that δ is a metric on $\mathcal{B}(X, Y)$.

(b) Give $\mathcal{B}(X, Y)$ the metric topology. Show that a sequence $\{f_n\}_{n\geq 1}$ in $\mathcal{B}(X, Y)$ converges to $g \in \mathcal{B}(X, Y)$ with respect to δ if and only if $\{f_n\}_{n\geq 1}$ converges uniformly to g.