DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
JANUARY 2011

Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correcct
PhD: 75% with at least three substantially correct.

1. Determine the values of α in the matrix below
$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 5 & 8 \\ 4 & 8 & \alpha \end{pmatrix}$$
for which the matrix is positive definite.

2. Find all the starting values x_0 for which the following one-point iteration method converges, and verify your answer.
$$x_{n+1} = e^{-x_n}.$$

3. Let $\{\psi_0(x), \psi_1(x), \ldots, \psi_n(x)\}$ be a basis for the vector space of all polynomials of degree less than or equal to n. For a real function $f(x)$, consider the problem of finding $p_n(x) = a_0\psi_0(x) + a_1\psi_1(x) + \cdots + a_n\psi_n(x)$, such that $p_n(x_i) = f(x_i), i = 0, 1, \ldots, n$. Prove that such polynomial $p_n(x)$ exists and is unique, if the points $\{x_i\}$ are distinct.

4. Given a vector norm $\|\cdot\|$ for the space $\mathbb{R}^n$, the induced matrix norm for an n-by-n matrix A is defined as
$$\|A\| = \max_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}.$$
For a non-singular real matrix A,
   (a) Show that the condition number $\kappa(A) = \|A\| \cdot \|A^{-1}\| \geq 1$.
   (b) Find $\kappa(A)$ for orthogonal matrix A, when the Euclidean norm is used.
   (c) Consider the linear system $Ax = b$ and its perturbed version $(A + \delta A)x = b + \delta b$. Show that
$$\frac{\|\delta b\|}{\|b\|} \leq \kappa(A)\frac{\|\delta A\|}{\|A\|}.$$

5. Consider the ordinary differential equation $y' = \frac{1}{2}y$. Give a third-order scheme (new or existing), and find the leading term of the local truncation error.

6. Find the coefficients $\{a_0, a_1, a_2, a_3\}$ and nodes $\{x_1, x_2\}$ so that the quadrature formula
$$\int_{-1}^{1} f(x)dx \approx a_0 f(-1) + a_1 f(x_1) + a_2 f(x_2) + a_3 f(1)$$
has the highest possible degree of precision, i.e., is exact for $\{1, x, x^2, \ldots, x^p\}$, where p is the largest value possible.

7. Consider the numerical differentiation formula

$$f'(x) \approx D_h f = af(x) + bf(x+h) + cf(x+2h).$$

(a) Find a, b and c so that the highest possible order is achieved.

(b) Bound the error in the approximation, assuming f is smooth enough.

(c) Use Richardson extrapolation to combine D_h and $D_{h/2}$ to produce a more accurate approximation.