UMASS AMHERST – BASIC EXAM – COMPLEX ANALYSIS

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Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.
NOTATION: We denote by $\mathbb{D}$ the open unit disc, i.e. $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$, and by γ its boundary, traversed once counterclockwise.

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(1) (a) Suppose f is an entire function whose real part $u(x, y) = \mathrm{Re}(f(x+iy))$ is a polynomial in x, y. Prove that $f(z)$ is a polynomial in z.
(b) Suppose f is an entire function whose real part $u(x, y) = \mathrm{Re}(f(x+iy))$ is bounded above. Prove that f is constant.

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(2) Suppose λ is a real number satisfying $\lambda > 1$ and $f(z) = ze^{\lambda - z} - 1$. Prove that $f(z)$ has a unique root in the unit disc $\mathbb{D}$ and that this root is a positive real number.

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(3) Suppose f is holomorphic on the region $A = \{z \in \mathbb{C} : 0 < |z| < 2\}$, and that for all $n \geq 0$,

$$\int_\gamma z^n f(z) dz = 0,$$

where γ is the unit circle traversed once counterclockwise. Show that f has a removable singularity at 0.

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(4) (a) For which z in $\mathbb{C}$ does $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{1 + z^{2n}}$ converge?
(b) At which z in $\mathbb{C}$ is the sum $f(z)$ of this series holomorphic?

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(5) Write down a conformal map that takes the "right-half" of the unit disc, namely $R = \{z \in \mathbb{D} \ : \ \mathrm{Re}(z) > 0\}$, **onto** the unit disc $\mathbb{D}$.

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(6) For each of the following statements, if the statement is true, give a proof; if it is false, demonstrate this by giving a counterexample.

(a) If f is holomorphic on a bounded connected open set $R \subset \mathbb{C}$ and has infinitely many zeros $z_1, z_2, z_3, \ldots$ in R, then f is identically 0 on R.

(b) If f, g are non-vanishing holomorphic functions on the open unit disc $\mathbb{D}$, satisfying

$$\frac{f'}{f}(1/n) = \frac{g'}{g}(1/n), \qquad n = 1, 2, 3, \ldots$$

then there exists a non-zero constant c such that $f(z) = c\, g(z)$ for all $z \in \mathbb{D}$.

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(7) Suppose f is holomorphic and bounded on the region

$$A = \{z \in \mathbb{C} \ : \ \frac{1}{2} < |z + i|\}$$

and is real on the real interval $(-1, 1) = \{z \in \mathbb{R}| -1 < z < 1\}$. State the Schwartz reflection principle and use it to prove that f is constant.

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(8) Let f be holomorphic on the open unit disk $\mathbb{D}$ and let d be the diameter of $f(\mathbb{D})$, that is, $d = \sup\big\{\, |f(z_1) - f(z_2)| : z_1, z_2 \in \mathbb{D} \,\big\}$. Prove that

$$\big|f\,'(0)\big| \leq \frac{1}{2}\, d.$$

Hint: Consider the function $g(z) = f(z) - f(-z)$.

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(9) Use contour integration to evaluate, for real $\alpha > 0$, the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha\, x)}{1 + x^2}\ dx.$$

Be sure to justify all your steps.

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(10) Calculate

$$\int_{\gamma} \frac{(z^2 - 1)^2}{z^2\,(z^2 + 4z + 1)} dz$$

where γ is the unit circle traversed once in the counterclockwise direction. Be sure to justify all your steps.
