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UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. Consider

$$\dot{x} = rx - \frac{x}{1+x^2}$$

   (a) What steady states exist and for what parameter values?

   (b) Draw the relevant bifurcation diagram denoting stable branches by a solid line and unstable by a dashed line.

   (c) Classify the types of bifurcation that occur.

2. Let $x(t)$ be the number of rabbits at time $t$ and $y(t)$ be the number of sheep at time $t$ governed by the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= x(3-x-y) \\ \frac{dy}{dt} &= y(2-x-y). \end{aligned}$$

   (a) Find the fixed points and investigate stability.

   (b) Draw the nullclines.

   (c) Sketch a plausible phase portrait.

3. Consider the system $\frac{d^2x}{dt^2} = x - x^2$.

   (a) Find all the equilibrium points and classify them.

(b) Find a conserved quantity.

(c) Using the conserved quantity sketch the phase portrait.

4. Prove the existence of a unique solution for the problem $u_t = ku_{xx}$, $u(x,0) = \phi(x)$, $u(0,t) = u(l,t) = 0$ in two ways:

   (a) using the maximum principle;

   (b) using the energy method.

(a) Find $u(x,t)$ that satisfies:

$$u_t = ku_{xx}$$

and also $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = x$.

(b) Apply Parseval's identity for the Fourier series decomposition of $f(x) = x$ (that you computed in part a) to determine

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

5. (a) Find a harmonic function $u(r,\theta)$ inside a wedge defined by the three sides: $\theta = 0$, $\theta = \beta$ and $r = a$, which satisfies the boundary conditions

$$u(r,\theta = 0) = 0, \quad u(r,\theta = \beta) = 0, \quad u(r = a,\theta) = h(\theta).$$

   (b) Apply this general solution to the special case of a semi-disk with $\beta = \pi$, $a = 1$ and $h(\theta) = 3\sin(\theta) - 7\sin(2\theta)$, to find $u(r,\theta)$ in this case.

6. Consider the PDE: $u_x + e^x u_y = 0$.

   (a) Find its general solution.

   (b) Plot the characteristic curves.

   (c) Find the solution that satisfies: $u(x \to -\infty, y) = \sin(y)$.