Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

January 15, 2010

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

For any topological spaces X and Y, we denote the set of continuous functions from X to Y by $C(X, Y)$.

(1) Let $\mathbb{R}, \mathbb{R}_d, \mathbb{R}_l, \mathbb{R}_{fc}$ denote the the real line with the standard topology, the discrete topology, the lower limit topology and the finite complement topology, respectively. Recall that a set $U \subset \mathbb{R}$ is open in $\mathbb{R}_{fc}$ if and only if $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite, and that a basis for the topology on $\mathbb{R}_l$ is the set of intervals $[a, b)$ where $a < b$.

Describe the following sets of functions: $C(\mathbb{R}, \mathbb{R}_d), C(\mathbb{R}_d, \mathbb{R}_l)$, and $C(\mathbb{R}_{fc}, \mathbb{R}_{fc})$.

(2) Consider the sequence $f_n(x) = \sin(nx)$ in $C(\mathbb{R}, \mathbb{R})$, where $\mathbb{R}$ is given the metric topology attached to the Euclidean metric. For which of the following topologies on $C(\mathbb{R}, \mathbb{R})$ does the sequence converge?
   (a) The uniform topology.
   (b) The topology of pointwise convergence (i.e. the point-open topology).
   (c) The compact-open topology (under our assumptions this topology coincides with the topology of compact convergence on $C(\mathbb{R}, \mathbb{R})$).

(3) Let $A \subset \mathbb{R}^2$ be the union of all lines through the origin with rational slope: $A = \bigcup_{m \in \mathbb{Q}} \{(y, my) \mid y \in \mathbb{R}\}$.
   (a) Is A connected?
   (b) Is A locally connected?

(4) (a) Define *locally compact.*
   (b) Prove that the rationals $\mathbb{Q}$ are not locally compact as a subspace of $\mathbb{R}$.

(5) (a) Let $p \colon X \to Y$ be a quotient map. If $p^{-1}(y)$ is connected for all $y \in Y$, and Y is connected, show that X is connected.
   (b) Give a counterexample to part (a) when p is not assumed to be a quotient map.

(6) Consider the space $\mathbb{R}^\omega$ of real-valued sequences. Let

$$S = \{(a_n) \in \mathbb{R}^\omega \mid \lim_{n \to \infty} a_n \text{ exists}\}.$$

Is S closed in the product topology? In the box topology?

(7) Let X, Y, Z be topological spaces, and let $f\colon X \times Y \to Z$ be a continuous function. If $U \subset Z$ is open and $K \subset Y$ is compact, show that

$$\{x \in X \mid f(x, y) \in U \text{ for all } y \in K\}$$

is open in X. Give an example to show that the condition that K be compact cannot be removed.