DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERICAL ANALYSIS EXAM
JANUARY 2010

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correcct

PhD: 75% with at least three substantially correct.

1. Consider $f(x) = 2 - x + x^2 - x^3$. Let $p_2(x)$ denote the second degree polynomial interpolation of $f(x)$ at $\{-1, 0, 1\}$.

   (a) Find $p_2(x)$.

   (b) Compute the L^∞ error of $p_2(x)$ on the domain $[-1, 1]$.

2. Suppose that the function $f(x)$ has enough regularity and a is a root of $f(x)$. Find the order of convergence of the Newton's method for the root a, assuming that the initial guess is sufficiently close to a. If the convergence is of order one, then give the rate of convergence(Hint: Make the appropriate assumptions on $f'(a)$, $f"(a)$, ...).

3. Find the Gauss-Lobatto like quadrature

$$\int_{-1}^{1} f(x)dx \approx \omega_1 f(-1) + \omega_2 f(x_0) + \omega_3 f(1)$$

   with the highest possible degree of precision.

4. For function $\sin(\pi x)$,

   (a) Find the value of a which solves the following optimization problem:

$$\min_a \int_{-1}^{1} (\sin(\pi x) - ax)^2 \, dx$$

   (b) Let $\hat{f}(x)$ be a polynomial with degree less than or equal to $n > 1$, which solves the minimization problem:

$$\min_{p(x) \in \mathbf{P_n(x)}} \int_{-1}^{1} (\sin(\pi x) - p(x))^2 \, dx$$

   Prove that $\hat{f}(x)$ is an odd function.

5. Consider the ordinary differential equation $\dfrac{dy}{dt} = 0.1y$

   (a) What is the order of the scheme: $y^{n+1} - y^n = 0.1 \Delta t y^n$? Derive the local truncation error.

   (b) Derive a 10th order scheme for the above equation.

6. (a) Write down the Jacobi and Gauss-Siedel methods for the system $Ax = b$ where $A =$

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 3 \end{pmatrix}$$

and $b =$

$$\begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}$$

(b) Prove or disprove that the Jacobi method for the system above converges for any initial guess.

7. Consider the fixed point iteration

$$x_{n+1} = \phi(x_n), \quad n = 0, 1, 2, ...$$

where $\phi(x) = Ax + Bx^2 + Cx^3$.

(a) Given a positive number α, determine the constants A, B, C such that the iteration converges locally to $1/\alpha$ with order 3(This will give a cubically convergent method for computing the reciprocal $1/\alpha$ which uses only addition, subtraction and multiplication).

(b) Determine the maximal possible interval in which the initial guess x_0 can lie in order that the iteration x_n converges to $\frac{1}{\alpha}$.