Department of Mathematics and Statistics
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# Basic Exam: Topology

January 20, 2009

**Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.**

**Passing standard:** For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

1. Let $A$ and $B$ be proper subsets of connected spaces $X$ and $Y$, respectively. Prove that the complement of $A \times B$ in $X \times Y$ is connected.

2. Let $X$ denote the "real line with two origins." As a set,

$$X = (\mathbb{R} \setminus \{0\}) \cup \{0^-, 0^+\}.$$

And the topology of $X$ has as a basis the collection $\mathcal{B}$ of those subsets $B$ of $X$ having one of the forms:

- $B \subset \mathbb{R} \setminus \{0\}$ and $B$ is open in $\mathbb{R}$; or
- $B = (U \setminus \{0\}) \cup \{0^+\}$ with $U$ an open neighborhood of 0 in $\mathbb{R}$; or
- $B = (U \setminus \{0\}) \cup \{0^-\}$ with $U$ an open neighborhood of 0 in $\mathbb{R}$.

   (a) Show that $\mathcal{B}$ is, in fact, a basis for a topology.

   (b) Show that $X$ (with the topology generated by $\mathcal{B}$) is not Hausdorff.

   (c) Show that the topology generated by $\mathcal{B}$ is not metrizable.

3. Let $G = \{\, (x, f(x)) : x \in X \,\}$ be the graph of a map $f\colon X \to Y$ from a compact space $X$ to a Hausdorff space $Y$. Prove that $G$ is closed in $X \times Y$ if and only if $f$ is continuous.

4. Let $(X, d)$ be a metric space. Fix $0 < \alpha < 1$. Consider some $f\colon X \to X$ such that for all $x, y \in X$,

$$d\big(f(x), f(y)\big) \leq \alpha\, d(x, y).$$

   (a) Prove $f$ is continuous.

   (b) Now assume $X$ is compact. Show that there exists a unique $x \in X$ such that $f(x) = x$.

5. Let $R^\omega$ be the set of all functions from the natural numbers $\mathbb{N}$ to $\mathbb{R}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ and let $f\colon \mathbb{R} \to R^\omega$ be defined by $f = (f_1, f_2, \dots)$, that is, $f(x) = \big(f_1(x), f_2(x), \dots\big)$ for each $x \in \mathbb{R}$. Prove or disprove each of the following:

   (a) $f$ is continuous in the product topology.

   (b) $f$ is continuous in the box topology.

6. Let $(X_i)_{i\in I}$ be a family of Hausdorff spaces indexed by some set $I$. Prove that the product $\prod_{i\in I} X_i$ (with its product topology) is locally compact if and only if each $X_i$ is locally compact and all but finitely many of the $X_i$ are compact.

7. The *suspension* $\Sigma(X)$ of a space $X$ is the quotient space $(X \times [-1,1])/\sim$ of the "cylinder" $X \times [-1,1]$ obtained by identifying its "top" $X \times \{1\}$ to a point and identifying its "bottom" $X \times \{-1\}$ to another point. In other words, $(x,t) \sim (y,s)$ if and only if: $(x,t) = (y,s)$ with $0 < t < 1$; or $t = s = 1$; or $t = s = -1$.

   Prove that the suspension $\Sigma(S^n)$ of the $n$-sphere $S^n$ is homeomorphic to the $(n+1)$-sphere $S^{n+1}$. (Recall that, for a nonnegative integer $k$, the $k$-sphere $S^k$ is the set of points in $\mathbb{R}^{k+1}$ of distance 1 from the origin.)