DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
MASTER'S OPTION EXAM — APPLIED MATH
January 2009

Do 5 of the following questions. Each question carries the same weight. Passing level is 60% and at least two questions substantially correct.

1. Consider the initial value problem

$$\frac{dx}{dt} = -x \log x\,, \qquad x(0) = x_0 > 0\,,$$

which is a simplified and normalized model of tumor growth.

(a) Determine the solutions of this initial value problem, giving them in an explicit analytic form.

(b) Find the stable equilibrium point $x^* > 0$, show that these solutions approach x^* as $t \to +\infty$, and that in fact

$$x(t) - x^* \approx (\log x_0) \cdot e^{-t} \qquad \text{for large } t$$

2. Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature 20. Suddenly, both of its ends are plunged into a bath of temperature 0. Write the differential equation, boundary conditions and initial condition and subsequently solve the problem, to find the temperature $u(x,t)$ at later times.

3. Consider the competing species model (in normalized variables):

$$\begin{aligned}\frac{dx_1}{dt} &= (\,1 - x_1 - \frac{1}{2}x_2\,)\,x_1 \\ \frac{dx_2}{dt} &= (\,1 - \frac{2}{3}x_1 - x_2\,)\,x_2\end{aligned}$$

(a) Find the equilibrium points in the quadrant $x_1, x_2 \geq 0$, and classify their types.

(b) Draw the phase portrait, with the nullclines. What is the predicted qualitative behavior of solutions in this model as time $t \to +\infty$?

4. Find the harmonic function $u(x, y)$ in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions

- $u_y = 0$ for $y = 0$ and $y = \pi$,
- $u = 0$ for $x = 0$ and
- $u = \sin^2(y)$ for $x = \pi$.

5. Consider the differential equations governing an undamped pendulum with applied torque: in terms of the angle $q = q(t)$ of the pendulum arm from the vertical direction, the DE is

$$\frac{d^2 q}{dt^2} + \alpha \sin q \;=\; \beta$$

for positive constants α and β.

(a) Briefly give a physical interpretation of the constants α and β.

(b) Determine the condition on α and β that is required to guarantee that equilibria exist for this dynamical system.

(c) In the case that equilibria exist, determine the type of these equilibria. Also, sketch the phase portrait of the system in this case, using the usual phase plane with coordinates $x = (q, \dot{q})$.

[HINT: Note first that stable and unstable equilibria come in pairs, and second that the phase portrait is symmetric with respect to the reflection $x_2 \to -x_2$.]

6. Consider the diffusion equations $u_t - ku_{xx} = f$ and $v_t - kv_{xx} = g$ with $f \leq g$ and $u \leq v$ for $x = 0$, $x = l$ and $t = 0$.

(a) Use an argument similar to that of the maximum principle proof, to establish that $u \leq v$ for $0 \leq x \leq l$ and $0 \leq t$.

(b) Assume that $v_t - v_{xx} \geq \sin(x)$ for $0 \leq x \leq \pi$ and $0 < t < \infty$, and also that $v(0, t) \geq 0$, $v(\pi, t) \geq 0$ and $v(x, 0) \geq 0$. Then, use part (a) to show that

$$v(x, t) \geq (1 - e^{-t}) \sin(x)$$