BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
JANUARY 2009

Do 7 of the following 9 problems.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Show your work!

Part I. Linear Algebra

1. Give an orthonormal basis (with respect to the usual Euclidean inner product) of the subspace of $\mathbf{R}^4$ defined by the system of equations

$$\begin{aligned} x_1 + x_2 - 3x_3 - x_4 &= 0 \\ x_1 - 3x_2 + x_3 - x_4 &= 0 \\ x_1 - x_2 - x_3 - x_4 &= 0. \end{aligned}$$

---

2. Let A be an $n \times n$ real matrix. Suppose A is idempotent, i.e. $A^2 = A$. Show that there exist vector subspaces $W_0, W_1 \subset \mathbf{R}^n$ such that

- $\mathbf{R}^n = \mathrm{Span}(W_0, W_1)$;
- $A(w_0) = \vec{0}$ for all $w_0 \in W_0$; and
- $A(w_1) = w_1$ for all $w_1 \in W_1$.

---

3. Fix a non-zero vector $\vec{x} \in \mathbf{R}^3$.

(a) It follows from elementary properties of **vector cross product** that

$$\vec{y} \mapsto \vec{x} \times \vec{y}$$

defines a linear transformation $\varphi_{\vec{x}} : \mathbf{R}^3 \to \mathbf{R}^3$ (you do not have to prove this). Determine the eigenvalues of $\varphi_{\vec{x}}$.

(b) View elements of $\mathbf{R}^3$ as column vectors; then the *matrix product*

$$\vec{y} \mapsto (\vec{y}^t)\vec{x}$$

defines a linear transformation $\phi_{\vec{x}} : \mathbf{R}^3 \to \mathbf{R}^9$ (you do not have to prove this). Determine its rank and its kernel (i.e. null space).

---

4. Let A be an $m \times n$ matrix, and B an $n \times k$ matrix. Show that

$$\mathrm{rank}(AB) \le \min(\mathrm{rank}(A), \mathrm{rank}(B)).$$

Part II. Advanced Calculus

1. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function. Suppose $\lim_{x\to\infty} f(x)/x = 0$ and suppose $\alpha := \lim_{x\to\infty} f'(x)$ exists and is finite. Show that $\alpha = 0$.

---

2. If the ellipse $\frac{x^2}{A} + \frac{y^2}{B} = 1$ is to enclose the circle $x^2 + y^2 = 2y$, what values of $A, B > 0$ minimize the area of the ellipse?

---

3. For which integers $p > 0$ is the function

$$f(x) := \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

differentiable? For which integers $p > 0$ is the derivative continuous? Justify your answer!

---

4. Consider the following sequence of functions on $\mathbf{R}$:

$$g_k(x) := \begin{cases} 1/k^2 & |x| \leq k \\ 1/x^2 & |x| > k. \end{cases}$$

Does the sum $g(x) := \sum_{k=1}^{\infty} g_k(x)$ converge uniformly on $\mathbf{R}$? Is $g(x)$ continuous on $\mathbf{R}$? Justify your answer!

---

5. Consider the vector field on $\mathbf{R}^3$

$$\vec{F}(x,y,z) = \frac{\vec{r}}{|\vec{r}|^3}$$

where, as usual, $\vec{r} = (x,y,z)$.

(1) Show that the $\vec{F}$ is divergence-free, in other words $\nabla \cdot \vec{F} = 0$.

(2) Show that

$$\iint_{S^2} \vec{F} \cdot d\vec{S} = 4\pi,$$

where S^2 is the unit sphere, $\{(x,y,z) : x^2 + y^2 + z^2 = 1\}$.

(3) Explain why these together do *not* violate Gauss' divergence theorem.
