DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
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Work all problems. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.

1. (**12** pts) Let $X_1, \ldots, X_n$ be a random sample from what is known as the Rayleigh distribution, where X_i has density

$$f(x_i;\theta) = \frac{x_i}{\theta^2} e^{-x_i^2/2\theta^2} I_{(0,\infty)}(x_i), \theta > 0.$$

You may use without proof, the fact that X_i^2/θ^2 follow a Chi-Square distribution with 2 degrees of freedom.

(a) Define **IN GENERAL** the meaning of a pivotal quantity. Then, find a pivotal quantity for θ, which uses all of $X_1, \ldots, X_n$.

(b) Use the pivotal quanty to develop an exact $100(1-\alpha)\%$ confidence interval for θ. Your result should involve a well known distribution.

2. (**54** pts) Let $X_1, \cdots, X_n$ be iid according to the Poisson distribution $Poisson(\lambda)$, i.e.,

$$P(X_i = k) = \frac{e^{(-\lambda)}\lambda^k}{k!}, \quad k = 0, 1, \cdots, \quad \lambda > 0.$$

(a) Let T be a statistic for θ, based on $X_1, \cdots, X_n$. Give a precise **definition** for T being sufficient for λ, and for T being complete for λ, respectively.

(b) Show that $\sum_{i=1}^n X_i$ is a complete and sufficient statistic for λ. (Note: You may use a well-known theorem to justify your answers.)

(c) Consider the parameter $\theta = e^{-\lambda}$. Define

$$Y_i = \begin{cases} 1 & X_i = 0 \\ 0 & X_i \neq 0 \end{cases} \qquad i = 1, \cdots, n.$$

Show that Y_i is an unbiased estimator for θ.

(d) Knowing that $\sum_{i=1}^n X_i$ follows a $Poisson(n\lambda)$ distribution, find $E(Y_i | \sum_{i=1}^n X_i)$.

(e) Explain why $\hat{\theta} = E(Y_i | \sum_{i=1}^n X_i)$ is the UMVUE for θ. State the name of the theorem if used.

(f) Again use the fact that $\sum_{i=1}^n X_i$ follows a $Poisson(n\lambda)$ distribution, find the mean and variance of $\hat{\theta}$.

(g) Find the Cramer-Rao lower bound for θ. How does it compare to the variance of $\hat{\theta}$?

(h) Find the MLE of θ, denoted by $\hat{\theta}_{MLE}$.

(i) What is the asymptotic distribution of $\hat{\theta}_{MLE}$? Use the distribution to construct a 95% confidence interval for θ.

3. (**12** pts) Let $X_1, \cdots, X_n$ be iid according to $N(0, \sigma^2)$. Let $\tau = 1/(2\sigma^2)$. Consider the prior for τ as the $Gamma(\alpha, \beta)$ distribution, i.e., with pdf

$$f(\tau|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\tau^{\alpha-1}e^{-\tau/\beta}, \quad \tau > 0, \quad \alpha > 0, \quad \beta > 0.$$

Note for the $Gamma(\alpha, \beta)$ distribution, the following results are known:

$$E(\tau) = \alpha\beta \qquad E(\frac{1}{\tau}) = \frac{1}{(\alpha-1)\beta} \quad \text{for } \alpha > 1.$$

(a) Find the posterior distribution of τ.

(b) Find the Bayes estimator of σ^2 under the squared error loss.

4. (**22** pts) Let $X_1, \cdots, X_n$ be iid according to $Exponential(\beta)$, i.e., the pdf of X_1 is

$$f(x|\beta) = \frac{1}{\beta}e^{-x/\beta}, \quad x > 0, \quad \beta > 0.$$

(a) Show that the joint Exponential distribution of $X_1, \cdots, X_n$, has a Monotone Likelihood Ratio (MLR) in $\sum_{i=1}^n X_i$.

(b) Find the UMP test for the hypothesis

$$H_0 : \beta \leq \beta_0 \qquad H_1 : \beta > \beta_0, \quad \text{where } \beta_0 > 0.$$

(c) Knowing that $\sum_{i=1}^n X_i$ follows the $Gamma(n, \beta)$ distribution, find the power function of the UMP test.