DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - PROBABILITY
WINTER 2008

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Suppose an individual can be either diseased (D) or healthy (H). A medical diagnostic test can say that an individual is diseased (+) or healthy (-). In a particular population,

   - the probability an individual is D and the test is + is 0.009,
   - the probability an individual is D and the test is - is 0.001,
   - the probability an individual is H and the test is + is 0.099,
   - and the probability an individual is H and test is - is 0.891.

   (a) (10 pts) Suppose an individual is randomly selected from that particular population. What is the probability that she is healthy?

   (b) (5 pts) Given that an individual is diseased, what the probability that the test is +?

   (c) (5 pts) Suppose an individual gets the test, and the test is positive. What is the probability that the individual is diseased?

2. Suppose $X \sim U(0, 1)$. Note that $f(x) = 1$ when $0 \leq x \leq 1$ and $f(x) = 0$ otherwise.

   (a) (5 pts) What is the probability that $\sqrt{X}$ is less than $1/2$?

   (b) (10 pts) What is the second moment of $1/X$?

   (c) (10 pts) Suppose that $G(y)$ is the cumulative distribution function (CDF) of a continuous random variable with probability distribution function (PDF) $g(y)$. Note that $G(y) = \int_{-\infty}^{y} g(t)dt$. Let $G^{-1}(p)$ be the inverse of the CDF. Let $Z = G^{-1}(X)$. Derive the PDF of Z.

3. Suppose Y has PDF $f(y) = c \exp(-y/2)$ when $y > 0$ and $f(y) = 0$ otherwise.

   (a) (5 pts) What is c?

   (b) (5 pts) Derive the moment generating function of Y.

   (c) (5 pts) Use the moment generating function to find $E(Y^k)$ for $k = 1, 2, 3$.

   (d) (5 pts) Prove that $E(Y^k) \geq E(Y)^k$ for $k > 1$. It is OK to cite a theorem.

   (e) (5 pts) What is the PDF of $3Y$?

(f) (5 pts) Let $X = Y1_{Y>3} - 3$ where $1_{Y>3} = 1$ if $Y > 3$ and 0 otherwise. Derive $E(X)$.

4. Suppose $X_i, i = 1, \ldots$ are independent and identically distributed with mean μ and variance $\sigma^2 < \infty$. Let $Z_i = (X_i - \mu)/\sigma$.

   (a) (5 pts) Let $M_n = n^{-1} \sum_{i=1}^{n} Z_i$. Prove that $\lim_{n\to\infty} Pr(|M_n| > 0) = 0$. For partial credit, you may just state a theorem.

   (b) (10 pts) Let $f(n)$ be a function of n. Find an $f(n)$ so that the variance of $f(n)M_n = 1$.

   (c) (10 pts) Let A be a constant. State and apply a theorem that will allow you to determine $\lim_{n\to\infty} Pr(f(n)M_n < A)$.