UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Numerical Analysis Exam
January 2008

1. In order to solve the equation

$$x^2 = 2^2,$$

one can use the following fixed-point methd:

$$x_{n+1} = \sqrt{2 + x_n}.$$

Determine an interval $[a, b]$ containing the root such for each x_0 in $[a, b]$, the iteration will converge to the root. Prove it.

2. Find the lowest order polynomial $p(x)$ which satisfies the conditions

$$p(0) = y_0, \quad p(0.1) = y_1, \quad p'(0) = v_0, \quad p'(0.1) = v_1,$$

for some values y_0, y_1, v_0, and v_1. Now, assuming that these values correspond to those of a given function $f(x) \in C^\infty([0, 0.1])$ and its derivative, given an estimate of the infinity-norm error over the interval $[0, 0.1]$ between $f(x)$ and $p(x)$.

3. Consider the Secant method for finding a root α of the function $f(x)$.

   (a) Use Newton Divided Differences to show that

   $$x_{k+1} - \alpha = \frac{f[x_k, x_{k-1}, \alpha]}{f[x_k, x_{k-1}]} \, (x_k - \alpha) \, (x_{k-1} - \alpha) \; .$$

   (b) Suppose

   $$M \equiv \frac{1}{2} \, \frac{\sup_{x_2} |f''(x_2)|}{\inf_{x_1} |f'(x_1)|} \; > 0 \; ,$$

   and show by induction that

   $$|x_k - \alpha| \leq \frac{z_k}{M} \; ,$$

   where the sequence z_k is defined by

   $$z_{k+1} = z_k \; z_{k-1}.$$

4. Consider the quadrature formula

$$I(f) = a_0 f(1/2) + a_1 f(t) \approx \int_{-1}^{1} f(x)\, dx,$$

where t is in $[-1, 1]$. Find a_0, a_1 and t such that the quadrature rule is exact for polynomials of the highest degree possible. Also, what is the degree?

5. Suppose A is a positive definite matrix: $x^T A x > 0$ for all $x \neq 0$. For $n \geq 2$ answer the following

   (a) Prove A is non-singular.

   (b) Is A a symmetric matrix? If yes, prove it. Otherwise, find a positive definite matrix which is not symmetric. (Be sure to show your example is indeed positive definite.)

6. For the ODE $y' = f(t, y)$ with solution $y(t)$ consider the linear multi-step numerical method

$$y_{n+1} = y_n + \frac{h}{12}(23 f(t_n, y_n) - 16 f(t_{n-1}, y_{n-1}) + 5 f(t_{n-2}, y_{n-2})),$$

where $n = 0, 1, \ldots$, and h is a uniform stepsize, i.e., $t_{n+1} - t_n = h$. Find a formula for the *local truncation error* and use it to determine the *global* order of the method.

7. An $n \times n$ complex matrix A is said to be *normal* if it commutes with its conjugate transpose, i.e. $AA^* = A^* A$, where $A^* = \bar{A}^T$.

   (a) Schur's Theorem states that any complex matrix is unitarily similar to an upper triangular matrix. Use Schur's theorem to prove the following *Spectral Theorem for Normal Matrices* for a given matrix A:

   A is normal $\iff$ A is unitarily similar to a diagonal matrix

   (b) Given *any* square matrix A it can be shown that

$$\lim_{k \to \infty} A^k = 0 \iff \rho(A) < 1.$$

   Here $\rho(A)$ is the *spectral radius* of A. Prove this result in the case where A is assumed to be normal.