BASIC EXAM – COMPLEX ANALYSIS

JANUARY 2008

Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.

Notation. For $w \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$, we let $B_r(w) = \{z \in \mathbb{C} : |z - w| < r\}$ be the open disc of radius r centered at w. We let $\mathbb{D} = B_1(0)$ be the open unit disc.

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1. Suppose f is an entire function and M, R are positive constants such that $|f(z)| \leq M$ on the circle $|z| = R$. Determine, with proof, a quantity N which depends only on k and M (not on f, r, R) such that for all $0 \leq r < R$,

$$|f^{(k)}(re^{i\theta})| \leq \frac{N}{(R-r)^k}, \qquad k = 0, 1, 2, \ldots.$$

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2. Suppose the coefficients of the power series $f(z) = \sum_{n \geq 0} a_n z^n$ satisfy the recurrence relation

$$a_0 = a_1 = 1; \qquad a_n - 7a_{n-1} + 12a_{n-2} = 0,\ n = 2, 3, 4, \ldots.$$

Determine the function f explicitly as well as the radius of convergence of the given power series.

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3. Prove that for a fixed complex number $w \in \mathbb{C}$,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{2w\cos(\theta)} d\theta = \sum_{n=0}^{\infty} \left(\frac{w^n}{n!}\right)^2.$$

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4. Let $s(y)$ and $t(y)$ be real differentiable functions of y on $-\infty < y < \infty$ satisfying $s(0) = 1, t(0) = 0$, with the property that the complex function

$$f(x + iy) = e^x (s(y) + it(y))$$

is entire. Determine $s(y)$ and $t(y)$ (with proof).

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5. Prove or disprove.

(1) The image of $\mathbb{C}$ under a non-constant entire function is dense in $\mathbb{C}$.

(2) If the radius of convergence of the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is R, then the radious of convergence of the series $\sum_{n=1}^{\infty} \frac{a_n}{n} (z - z_0)^n$ is also R.

(3) If $f(z)$ is analytic in a domain D and and $u(x, y) = Re(f(x+iy))$, then the function $H(x, y) = \sin(u(x, y))$ is harmonic in D.

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6. (a) State, but do not prove, the Schwartz Lemma.

(b) Let f be a holomorphic function from the open unit disc D into itself. Prove the inequality

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2},$$

for all $z \in D$.

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7. Using a contour integral, compute

$$\int_0^\infty \frac{\cos(x)-1}{x^2} dx.$$

Justify all estimates.

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8. Let f be a one-to-one holomorphic map from an (open, connected) region D_1 onto a region D_2. Suppose that D_1 contains the closure of the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Prove that for $w \in f(\mathbb{D})$, the inverse function $f^{-1}(w)$ is given by

$$f^{-1}(w) \quad = \quad \frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)-w} \cdot z dz,$$

where the circle $|z| = 1$ is traversed once counterclockwise.

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9. (a) Compute

$$\int_C \frac{dz}{(z-i)^2 \cos(z)},$$

where C denotes the circle $\{z : |z| = 4\}$ traversed once counterclockwise.

(b) Let C be the circle $\{z : |z| = 2\}$ traversed once counterclockwise. Compute the integral $\int_C \frac{z^2(5z-1)^8}{1-z^{10}} dz$.

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10. Suppose f is holomorphic in a connected and simply connected open region Ω and satisfies $|f(z)-1| < 1$ for all $z \in \Omega$. Suppose C is a closed contour in Ω traversed once counterclockwise. Determine, with proof, the value of the integral

$$\int_C \frac{f'(z)}{f(z)} dz.$$
