BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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- Do 7 of the following 9 problems.
- **Passing Standard:**
  - For Master's level: 60% with three questions essentially complete (including at least one from each part)
  - For Ph. D. level: 75% with two questions from each part essentially complete.
- Show your work!

Part I. Linear Algebra

1. Denote by $D = \frac{d}{dx}$ the differential operator on the set $\mathbf{P}$ of all real, one-variable polynomials of degree ≤ 3 (including the zero polynomial). It is a fact that $\mathbf{P}$ is a real vector space and that D is a linear transformation from $\mathbf{P}$ to itself. Determine the characteristic polynomial and the minimal polynomial of this operator D.

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2. Let A be a complex $n \times n$ matrix for which $A^3 = A$. Prove that $\text{rank}(A) = \text{trace}(A^2)$.

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3. We say that a real, $n \times n$ symmetric matrix A is *positive definite* if $(A\vec{x}) \cdot \vec{x} > 0$ for all non-zero $\vec{x} \in \mathbf{R}^n$, where $\cdot$ denotes the usual inner product on $\mathbf{R}^n$.

(a) Show that A is positive definite if and only if all of its eigenvalues are positive.

(b) If A is positive definite, show that there exists another positive definite matrix B such that $A = B^2$.

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4. For a vector subspace W of $\mathbf{R}^n$, the orthogonal complement of W is defined by

$$W^\perp := \{\vec{x} \in \mathbf{R}^n : \vec{x} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W\},$$

where $\cdot$ denotes the usual inner product on $\mathbf{R}^n$. Show that $(W^\perp)^\perp = W$ for every subspace W of $\mathbf{R}^n$.

Part II. Advanced Calculus

1. Prove *directly* the following special case of the Arithmetic-Geometric Mean inequality: For any integer $n \geq 1$, if $y_1, \ldots, y_n$ are positive real numbers with product 1, then $y_1 + \cdots + y_n \geq n$.

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2. For each integer $n \geq 1$, let $f_n(x) = n^2 x^n (1-x)$.
(a) Show that this sequence of functions converges *pointwise* on $[0,1]$.
(b) Does this sequence of functions converges *uniformly* on $[0,1]$?
(c) Does $\lim_{n\to\infty} \int_0^1 f_n(x)dx \stackrel{?}{=} \int_0^1 \Big(\lim_{n\to\infty} f_n(x)\Big) dx$?

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3. Let $f(x)$ be a function which is continuously differentiable on the closed interval $[a,b]$. If $f(x)$ is not linear, show that there exists a number $c \in (a,b)$ at which

$$|f'(c)| > \left|\frac{f(b)-f(a)}{b-a}\right|.$$

(note: this is *not* the mean-value theorem!)

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4. Calculate

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS,$$

where S is the surface

$$S = \{(x,y,z) : x^2+y^2 = 1, -1 \leq z \leq 0\} \cup \{(x,y,z) : x^2+y^2 \leq 1, z = -1\},$$

oriented by its *outward*-point normal vector $\vec{n}$, and

$$\vec{F}(x + \vec{\imath} + y\vec{\jmath} + z\vec{k}) = (y + e^{xz})\vec{\imath} - (x + e^{yz})\vec{\jmath} + (e^{xyz})\vec{k}.$$

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5. Suppose $f(x)$ is Riemann-integrable on $[0,1]$ with $0 < f(x) < 1$. Show that $\int_0^1 f(x)dx > 0$.
