Your Name :

Department of Mathematics and Statistics
University of Massachusetts Amherst

Advanced Qualifying Exam– Differential Equations.

January 25th, 2008

This exam consists of seven (7) problems all carrying equal weight. You must do five (5) of them. Passing level: 75% with at least three (3) substantially complete solutions. Please **justify** all your steps properly by indicating (or stating) the result you are using. Please write each problem clearly and neatly in a separate page.

(1) Let $f(x)$ be a smooth vector field on $\mathbb{R}^n$. Suppose that the maximal interval of existence of the solution $x(t)$ of an initial value problem

$$x' = f(x), \quad x(0) = x_0 \in \mathbb{R}^n$$

is $a < t < b$, where $0 < b < \infty$. Prove that if K is any compact subset of $\mathbb{R}^n$, then there exists a sequence $t_n \to b$ with $t_n < b$ such that $x(t_n) \notin K$.

(**2**) Consider the system of ODEs

$$\begin{cases} x' &= x - x^2 + y \\ y' &= b\,x - y, \end{cases} \tag{1}$$

where b is a positive constant.

Prove that there exists a solution $(x(t), y(t))$ of (1) satisfying

$$\lim_{t\to-\infty}(x(t), y(t)) = (0, 0), \qquad \lim_{t\to+\infty}(x(t), y(t)) = (\,b+1, b(b+1)\,).$$

(**3**) Let $\varphi \in C^1(\mathbb{R})$ with compact support and consider the real-valued function u on the upper half-plane $\mathbb{R}^2_+ = \{x = (x_1, x_2) \, : x_2 > 0\}$ defined by

$$u(x_1, x_2) := \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{\varphi(y)}{(x_1 - y)^2 + x_2^2} \, dy$$

(a) What PDE and type of problem does u satisfy on the upper half-plane? (Be precise and explain your answer.)

(b) Prove that for each $x = (x_1, x_2) \in \mathbb{R}^2_+$,

$$1 = \int_{\mathbb{R}} K(x, y) \, dy$$

where $K(x, y) = \frac{x_2}{\pi} \frac{1}{|x-y|^2}, \; y \in \mathbb{R} = \partial \mathbb{R}^2_+$.

(c) Use **(b)** to prove rigorously that for each $x^0 \in \mathbb{R} = \partial \mathbb{R}^2_+$,

$$\lim_{x \to x^0, \, x \in \mathbb{R}^2_+} u(x) \; = \; \varphi(x^0).$$

(Hint: Note that by hypothesis φ is bounded and uniformly continuous.)

(4) Suppose that $p(u)$ is a smooth, real-valued function of $u \in \mathbb{R}^n$ such that $p(u) \to \infty$ as $|u| \to \infty$, and such that the gradient of p, $\nabla p(u)$, vanishes at exactly N distinct points, $c_1, \ldots, c_N$, where $N > 1$. Suppose that $p(c_1) < \cdots < p(c_N)$, and in addition that the Hessian matrix, $\nabla^2 p(u)$ at $u = c_N$ has exactly one negative eigenvalue $\lambda_1 < 0$ and $n-1$ positive eigenvalues $\lambda_j > 0, \;\; 2 \le j \le n$.

Prove that there there is a solution $u(t)$ of the gradient system

$$u' = -\nabla p(u),$$

that satisfies the limiting conditions

$$\lim_{t \to -\infty} u(t) = c_N, \quad \lim_{t \to +\infty} u(t) = c_k,$$

for some critical point c_k with $k \le N - 1$.

(**5**) Let u be the solution to the homogeneous wave equation

$$\partial_{tt}u - \Delta\, u \;=\; 0, \qquad \text{on } \mathbb{R}^{n+1} \qquad u(x,0) = g(x), \quad \partial_t\, u(x,0) \;=\; h(x),$$

where g and h are in $\mathcal{S}(\mathbb{R}^n)$, the space of Schwartz functions.

(a) Use the Fourier transform to find an expression for $\widehat{u}(\xi,t),\;\; \xi \in \mathbb{R}^n$.

(b) Use **(a)**, the properties of the Fourier transform and the characterization of the Sobolev spaces $H^s(\mathbb{R}^n)$ via the Fourier transform to prove that for any fixed $s \geq 0$

$$\|u(,t)\|_{H^s(\mathbb{R}^n)} \leq \text{const.}\Big(\, \|g\|_{H^s(\mathbb{R}^n)} + (1+t)\,\|h\|_{H^{s-1}(\mathbb{R}^n)}\Big)$$

for all $t > 0$. (Hint: Do not attempt to find $u(x,t)$ but rather work with $\widehat{u}(\xi,t)$.)

(**6**) Let $I = (0,1)$ and let $u : \bar{I} \times [0,T]$ be a smooth solution to the the mixed initial/boundary value problem

$$(1) \qquad \begin{cases} u_{tt} - u_{xx} + \alpha u_t \;=\; 0 & \text{on } \; I \times (0,T] \\ u \equiv 0 \qquad \text{on } \{x=0\} \times [0,T] \cup \{x=1\} \times [0,T] \\ u = g, \quad \text{and} \quad \partial_t u = h & \text{on } I \times \{t=0\} \end{cases}$$

where g, $h \in C_c^\infty(I)$ (smooth and compactly supported functions), and α is a **positive** constant.

Let $E[u] := \frac{1}{2}\int_0^1 |u_t|^2 + |u_x|^2\, dx$ be the ‘energy’ associated to (1) where the integrand is understood to be evaluated at (x,t).

(a) Prove that $E(t) \leq E(0)$ for all $t \in (0,T]$.

(b) Prove the uniqueness of classical solutions to (1).

(**7**) Let $a \in \mathbb{R}$, $a \neq 0$. Let δ be the Dirac delta distribution and $H(x)$ be the Heavside function $H(x) = 1$ if $x > 0$ and 0 if $x \leq 0$.

(a) Prove that $e^{-ax}\delta = \delta$ and find H' — both understood in the sense of distributions.

(b) Find the fundamental solution for

$$(1) \qquad L = \frac{d}{dx} - a \qquad \text{on } \mathbb{R};$$

that is, find the solution in the sense of distributions of $\frac{du}{dx} - a\,u = \delta$ and check your answer indeed satisfies (1) in de sense of distributions. (Hint. consider e^{-ax} as integrating factor).

(c) Let $L = -\frac{d^2}{dx^2} - a^2$ on $\mathbb{R}$ and let $u(x) = \frac{1}{a}\sinh(a\,x)$ if $x > 0$ and 0 if $x \leq 0$. Prove that $Lu = \delta$ in the sense of distributions.

(Hint. Recall that $\sinh x = 1/2\,(e^x - e^{-x})$ and $\cosh x = 1/2(e^x + e^{-x})$.)