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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.

2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$

1. Let X be an arbitrary nonempty set, $\mathcal{A}$ an algebra of subsets of X, $\mathcal{A}_\sigma$ the class of all countable unions of sets in $\mathcal{A}$, and μ a premeasure on $\mathcal{A}$. For any subset E of X, define

$$\mu^*(E) = \inf\left\{ \sum_{j=1}^{\infty} \mu(A_j) : A_j \in \mathcal{A} \text{ and } E \subset \bigcup_{j=1}^{\infty} A_j \right\}.$$

Also define $\mathcal{M}^*$ to be the class of subsets E of X satisfying

$$\mu^*(G) = \mu^*(G \cap E) + \mu^*(G \cap E^c) \text{ for all } G \subset X.$$

According to Carathéodory's Theorem, $\mathcal{M}^*$ is a σ-algebra containing the algebra $\mathcal{A}$, and μ^* is a measure on $\mathcal{M}^*$ that equals μ on $\mathcal{A}$.

(a) Let E be a set in $\mathcal{M}^*$ satisfying $\mu^*(E) < \infty$. Prove that for any $\varepsilon > 0$ there exists a set $A \in \mathcal{A}_\sigma$ such that $E \subset A$ and $\mu^*(A \setminus E) < \varepsilon/2$.

(b) Let E be a set in $\mathcal{M}^*$ satisfying $\mu^*(E) < \infty$. Prove that for any $\varepsilon > 0$ there exists a set $B \in \mathcal{A}$ such that $\mu^*(B \bigtriangleup E) < \varepsilon$. Recall that $B \bigtriangleup E = (B \setminus E) \cup (E \setminus B)$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Let f be a nonnegative, μ-integrable function mapping X into $[0, \infty)$. Prove that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E f \, d\mu < \varepsilon$ for any set $E \in \mathcal{M}$ satisfying $\mu(E) < \delta$. (**Hint.** Approximate f by a suitable bounded function.)

   (b) Assume that the measure μ on $(X, \mathcal{M})$ is σ-finite. Let ν be a finite measure on $(X, \mathcal{M})$. Prove that $\nu \ll \mu$ is equivalent to the following: for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\nu(E) < \varepsilon$ for any set $E \in \mathcal{M}$ satisfying $\mu(E) < \delta$. (**Hint.** Use part (a) to prove one o

   For $n \in \mathbb{N}$ consider the partition $t_0 < t_1 < \ldots < t_{2^n - 1}$ of the interval $[0, 1)$ with $t_j = j/2^n$. Define the functions

$$r_n(t) = (-1)^j \text{ if } j/2^n \leq t < (j+1)/2^n, j = 0, 1, \ldots, 2^n - 1.$$

   Prove that if $f \in L^1([0, 1), m)$, then

$$\lim_{n \to \infty} \int_{[0,1)} f(t) r_n(t) \, dt = 0.$$

   (**Hint.** First consider $f = 1_{[a,b]}$ for $[a, b] \subset [0, 1)$.)

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(**Hint.** First consider $f = 1_{[a,b]}$ for $[a, b] \subset [0, 1)$.)

4. For $t > 0$ and $x > 0$ define

$$f(x,t) = \frac{e^{-x} - e^{-xt}}{x} \quad \text{and} \quad F(t) = \int_0^\infty f(x,t)\,dx.$$

Prove that for all $t > 0$, $F(t) = \log(t)$. (**Hint.** Consider dF/dt.)

5. For each $j = 1, 2$, let $(X_j, \mathcal{M}_j)$ be a measurable space and let μ_j and ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$.

   (a) For $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$ define

$$\alpha(E) = \int_E \left(\frac{d\nu_1}{d\mu_1}(x_1) \cdot \frac{d\nu_2}{d\mu_2}(x_2) \right) d(\mu_1 \times \mu_2)(x_1, x_2).$$

   Prove that α is a measure on $\mathcal{M}_1 \otimes \mathcal{M}_2$ and that $\alpha = \mu_1 \times \mu_2$.

   (b) Prove that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and that

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \nu_1)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \cdot \frac{d\nu_2}{d\mu_2}(x_2).$$

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f be a measurable function mapping X into $\mathbb{R}$ and let $\{f_n, n \in \mathbb{N}\}$ be a sequence of measurable functions mapping X into $\mathbb{R}$.

(a) Assume that for all $\delta > 0$

$$\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \delta\}) < \infty.$$

Prove that $f_n \to f$ a.e. (**Hint.** Let $A = \{x \in X : f_n(x) \to f(x)\}$ and find an appropriate upper bound for $\mu(A^c)$.)

(b) Assume that $f_n \to f$ in measure. Prove that there exists a subsequence f_{n_j} that converges to f a.e. (**Hint.** Use part (a).)

7. Consider the map T on $L^1([0,1],m)$ defined by $Tf(x) = \int_0^x f(t)\,dt$ for $f \in L^1([0,1],m)$ and $x \in [0,1]$.

   a) Prove that for any $n \in \mathbb{N}$

$$T^n f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\, f(t)\,dt.$$

   (b) Prove that T maps $L^1([0,1])$ into $L^1([0,1])$ and is a bounded, linear operator. Recall that $\|T\| = \sup\{\|Tf\| : f \in L^1([0,1]), \|f\| = 1\}$, where $\|\cdot\|$ denotes the norm on $L^1([0,1])$.

   (c) Prove that $\|T^n\| \leq 1/n!$.

8. Let $\mathcal{H} = L^2([-1,1])$, which denotes the real Hilbert space consisting of all square integrable functions mapping X into $\mathbb{R}$ and equipped with the usual inner product and norm.

(a) Use the Gram-Schmidt process to find an orthonormal sequence $\{\varphi_0, \varphi_1, \varphi_2\}$ in $\mathcal{H}$ whose linear span equals the linear span of $\{1, x, x^2\}$.

(b) By using an appropriate theorem or appropriate theorems about Hilbert spaces, determine

$$\min_{a\in\mathbb{R}, b\in\mathbb{R}} \int_{-1}^{1} |x^2 - a - bx|^2 \, dx.$$

Indicate what theorem(s) about Hilbert space you are using in your answer.

(c) By using an appropriate theorem or appropriate theorems about Hilbert spaces, determine

$$\max \int_{-1}^{1} x^2 f(x) \, dx.$$

where f is subject to the restrictions

$$\int_{-1}^{1} f(x) \, dx = 0, \int_{-1}^{1} x f(x) \, dx = 0, \int_{-1}^{1} |f(x)|^2 \, dx = 1.$$

Indicate what theorem(s) about Hilbert space you are using in your answer. (**Hint.** Define $\mathcal{M}$ to be the linear span of $\{\varphi_0, \varphi_1, \varphi_2\}$. Write $f = g + h$, where g is the orthogonal projection of f onto $\mathcal{M}$ and h is the orthogonal projection of f onto $\mathcal{M}^{\perp}$.)