University of Massachusetts
Dept. of Mathematics and Statistics
Basic Exam - Topology
January 27, 2007

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing Standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

1. Let X, Y be topological spaces with X compact. Let $b \in Y$ and let $U \subset X \times Y$ be an open subset which contains $X \times \{b\}$. Show that there exists a neighborhood V of b such that $X \times V \subset U$.

2. Let $A \subset \mathbb{R}$ be compact, and let $B \subset \mathbb{R}$ be closed.

   (a) Show that the set
   $$C = \{a + b \mid a \in A, b \in B\}$$
   is closed.

   (b) Find an example of closed sets A and B in $\mathbb{R}$ for which $C \subset \mathbb{R}^2$ is not closed.

3. Let A and B be proper subsets of spaces X and Y, respectively. If X and Y are both connected, show that $(X \times Y) - (A \times B)$ is connected. (Hint: Try looking at the case $X = Y = [0, 1],\ A = B = (0, 1)$).

4. Let $X = (0, 1]$, $Z = [0, 1)$ and $Y = [0, 1]$. Let $f_n : Y \to Y$ be the continuous function whose graph consists of segments from $(0, 1)$ to $(\frac{1}{n}, 0)$ and from $(\frac{1}{n}, 0)$ to $(0, 1)$. Let $g_n : X \to Y$ be the restriction of f_n to $(0, 1] = X$. Let $h_n : Z \to Y$ be the restriction of f_n to $[0, 1) = Z$.

   (a) Does the sequence h_n converge for the uniform metric D on the set $C(X, Y)$ of continuous functions from X to Y. (Here, $D(f, g) = \sup_{x \in X}\ |g(x) - f(x)|$).

   (b) Does the sequence g_n converge pointwise?

   (c) Does the sequence g_n converge in the compact open topology on $C(X, Y)$?

   (d) Does the sequence h_n converge in the compact open topology on $C(Z, Y)$?

5. Let (M, d) be a metric space and suppose K and H are subsets of M. For $x \in M$, define $d(x, k) = \inf_{y \in K} d(x, y)$ and define $d(H, K) = \inf_{x \in H} d(x, K)$.

   (a) Prove that if K is closed and H is compact, then $d(H, K) = 0$ if and only if $H \cap K \neq \emptyset$.

(b) Show by the way of an example that if K and H are closed in M, then it is possible for $H \cap K = \emptyset$ and $d(H, K) = 0$. (Hint: Find and example where $M = \mathbb{R}^2$.)

6. Let $X = \mathbb{R}^n / \sim$ be the quotient of $\mathbb{R}^n$ by the equivalence relation: $x \sim y$ if the difference vector $x - y$ has integer coordinates. Show that:

   (a) X is connected.

   (b) X is compact.

   (c) X is Hausdorff.

7. Let X be a metric space.

   (a) Show that X has a countable dense subset if and only if X has a countable base for its topology.

   (b) Suppose that X is compact. Show that both conditions from part (a) hold (you only need to show one!).