Department of Mathematics and Statistics
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Basic Exam - Complex Analysis

January 2007

Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. Prove that there does not exist a one-to-one conformal map from the punctured unit disc $\{z : 0 < |z| < 1\}$ onto the annulus $A = \{z : 1 < |z| < 2\}$.

2. Find a one-to-one conformal map from the region $\{z \ : \ 1 < |z| < 2, \text{ and } \mathrm{Re}(z) > 0\}$ onto the rectangle $\{x + iy \ : \ 0 < x < \pi \text{ and } 0 < y < \ln(2)\}$

3. State and prove the Swartz Lemma.

4. (a) Find the Laurent series expansion of the function $f(z) = \dfrac{1}{z^2 - 4z + 3}$ valid near and centered at $z_0 = 1$. For what values of z does the series converges?

   (b) Find the radius of convergence R of the Taylor series about $z = 1$ of the function
$$f(z) = \frac{1}{1 + z^2 + z^4 + z^6 + z^8 + z^{10}}.$$
   Express the answer explicitly as a real number.

5. Let $f(z)$ be an analytic function on the punctured complex plane $\mathbb{C} \setminus \{0\}$, satisfying
$$|f(z)| \quad \geq \quad \frac{1}{|z|^d},$$
   for some real number d. Show that d must be an integer and there exists a constant $c \in \mathbb{C}$, such that $f(z) = cz^{-d}$.
   Hint: Reduce to the case $0 < d \leq 1$ *and analyze the singularities of* f.

6. Prove that every one-to-one holomorphic map f from the upper-half-plane $\mathbb{H} := \{x + iy \ : \ x, y \in \mathbb{R}, \ y > 0\}$ onto itself is a fractional linear transformation with real coefficients and positive determinant. That is, f can be written in the form:
$$f(z) \quad = \quad \frac{az + b}{cz + d},$$
   where $a, b, c, d \in \mathbb{R}$, and $ad - bc = 1$.

7. (a) Prove that the series
$$\sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}$$
   defines a meromorphic function $f(z)$, periodic with period 1, over the complex plane.

(b) Prove that the function $g(z) := f(z) - \dfrac{\pi^2}{\sin^2(\pi z)}$ is an entire function.

8. Evaluate the following integrals

   (a) $\displaystyle\int_C \frac{\cos(z)dz}{z^2(z^5-1)}$, where C is the circle $\{|z| = \frac{1}{2}\}$.

   (b) $\displaystyle\int_C \frac{z^4\cos(1/z)}{z^5+1}dz$ where C is the circle $\{|z| = 3\}$.

9. Evaluate the integral $\displaystyle\int_0^\infty \frac{\cos(x)dx}{x^2+4}$. Justify all your steps!!!

10. Let f be a non-constant entire function and $C := \{z : |z| = 1\}$ the unit circle. Suppose $|f(z)| = 1$, for all $z \in C$. Prove that the winding number $W(f(C), 0) := \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f'(z)dz}{f(z)}$ is positive.