Advanced Calculus/Linear Algebra Basic Exam
January 2007

Do 7 of the following 9 problems. Indicate clearly on your answer booklet which problems should be graded.

Passing standard: For Master's level, 60% with three questions essentially correct (including at least one from each part). For Ph.D. level, 75% with two questions from each part essentially complete.

Part I: Linear algebra

1. (a) Let A be an 8×8 complex matrix with characteristic polynomial

$$p_A(x) = (x-1)^4(x+2)^2(x^2+1)$$

and minimal polynomial

$$m_A(x) = (x-1)^2(x+2)^2(x^2+1).$$

Determine all possible Jordan canonical forms of A.

(b) What is the dimension of the eigenspace for $\lambda = 1$ in each case?

2. Let $V = \mathbf{R}^3$ (regarded as a space of column vectors) and consider the bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbf{R}$$

given by

$$\langle v_1, v_2 \rangle = v_1^t \cdot \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \cdot v_2$$

with v_1^t the transpose of v_1. Find an orthogonal basis for V with respect to this pairing. Does V have an orthonormal basis with respect to this pairing?

3. Let $V = \mathbf{C}^2$ and let $T : V \to V$ be the linear transformation with matrix

$$\begin{bmatrix} 14 & -16 \\ 9 & -10 \end{bmatrix}.$$

Find all T-invariant subspaces of V: that is, find all subspaces $W \subseteq V$ such that $T(W) \subseteq W$.

4. (a) Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a linear transformation. Suppose that $V = \operatorname{im} T + \ker T$; that is, V is spanned by the image and kernel of T. Prove that V is then the direct sum of $\operatorname{im} T$ and $\ker T$.

(b) Give a counterexample to the above assertion when V is infinite dimensional.

Part II: Advanced calculus

1. Find the value $\alpha \in \mathbf{R}$ which minimizes

$$\int_{C_\alpha} (y^2+1)dx + xdy$$

where C_α is the arc of the curve $y = \alpha x(1-x)$ from $(0,0)$ to $(1,0)$.

2. Let $f(x) = |x|^3$. Prove that $f'(x)$ and $f''(x)$ exist for all real x, while $f'''(0)$ does not exist.

3. Let $f(x)$ be a real differentiable function defined for $x \geq 1$ such that

$$f(1) = 1; \qquad f'(x) = \frac{1}{x^2 + f(x)^2}.$$

Show that

$$\lim_{x\to\infty} f(x)$$

exists and is less than $1 + \frac{\pi}{4}$.

4. Evaluate

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right) e^{-2x} dx.$$

5. Find the point on the intersection of the parabaloid $z = x^2 + y^2$ and the plane $x + y + 2z = 2$ which is closest to the origin.