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ADVANCED EXAM — DIFFERENTIAL EQUATIONS
JANUARY 2007

Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1a.) Show that

$$E(x,y) = \frac{1}{2}y^2 - \cos x$$

is non-increasing on all solutions $(x(t), y(t))$ of

$$(*) \quad \begin{aligned} x' &= y \\ y' &= -cy - \sin x, \end{aligned}$$

for any given constant $c \geq 0$.

1b.) Describe the structure of the global phase plane of $(*)$ in the region

$$\{(x,y) : -\frac{3\pi}{2} < x < \frac{3\pi}{2}\}$$

when

$$(1) \quad c = 0, \qquad\qquad (2) \quad c = 3.$$

Complete answers should include sketches depicting all periodic and/or connecting orbits (if any), and the local behavior of all solutions near rest points (in each case), and should be supported by accompanying analytical calculations and arguments.

2a.) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are smooth and that a bounded domain $\sigma \subset \mathbb{R}^n$ is defined by $\sigma = \{x : g(x) < 0\}$. Suppose that there is a constant $\delta > 0$ with $\nabla g(x) \cdot f(x) < -\delta$ for all $x \in \partial\sigma$. Prove that if $x(t)$ is the solution to $x' = f(x), x(0) = x_0$, then $x_0 \in \sigma$ implies that $x(t) \in \sigma$ for all $t \geq 0$.

2b.) Consider the nonautonomous initial value problem

$$\frac{dx}{dt} = f(x,t), \quad x(0) = x_0.$$

Suppose that a tube-like region Σ in $\mathbb{R}^n \times \mathbb{R}$ is defined as the union (over $t \in \mathbb{R}$) of bounded domains

$$\sigma_t = \{x \in \mathbb{R}^n \ : \ G(x,t) < 0\}$$

for some smooth function $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Formulate a condition under which $(x_0, 0) \in \Sigma$ implies that $(x(t), t) \in \Sigma$ for all $t \geq 0$.

3.) Let $\Omega \subseteq \mathbb{R}^n$ be a smooth, bounded domain, and for every $\tau \geq 0$ let $w = w(x,t;\tau)$ denote the solution of the initial-value problem:

$$\begin{cases} w_t - \Delta w = 0 & x \in \Omega, \quad t > \tau \\ \qquad w = 0 & x \in \partial\Omega, \quad t > \tau \\ w|_{t=\tau} = f(x,\tau) & x \in \Omega, \end{cases}$$

where $f(x,t)$ is a given function defined and smooth for $x \in \Omega, t \geq 0$. Express the solution $u = u(x,t)$ of the problem

$$\begin{cases} u_t - \Delta u = f(x,t) & x \in \Omega, \quad t > 0 \\ \qquad u = 0 & x \in \partial\Omega, \quad t > 0 \\ u|_{t=0} = 0 & x \in \Omega, \end{cases}$$

in terms of w and fully justify this expression.

HINT: To infer the required expression consider the analogous ODE system $\dot{x} = Ax + f(t)$ *with* $x(0) = 0$.

4.) Consider the equation for a vibrating string with "internal damping" (involving the rather unusual u_{xxt} term):

$$(*) \qquad \begin{cases} u_{tt} = u_{xx} + \epsilon^2 u_{xxt} & in\ 0 < x < 1, \ \ t > 0 \\ u(0,t) = 0 = u(1,t) \end{cases}$$

(i) Show that any solution of (*) satisfies

$$\frac{dE}{dt} \leq 0 \ \ with\ E(t) = \frac{1}{2}\int_0^1 (u_t^2 + u_x^2)\, dx.$$

(ii) Use the result (i) to deduce the uniqueness of the solution of the initial value problem for (*).

5.) Let $\Omega_a = \{(x,y) \ : \ 0 < x < a, 0 < y < 1\}$.

(i) Find the smallest number $a > 0$ such that the problem

$$(*) \qquad \begin{cases} u_{xx} + u_{yy} + 13u = f & in\ \ \Omega_a \\ u = 0 \quad on\ \ \partial\Omega_a \end{cases}$$

can have more than one solution for *some* function $f = f(x,y)$.

(ii) For the value of a found in part (i) discuss solving (*) when $f(x,y) = \sin \pi y$.

6.) State and prove the classical maximum principle for the initial boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0 & for\ \ (x,t) \in \Omega \times (0,T) \\ u(x,t) = \varphi(x,t) & for\ x \in \partial\Omega, \ \ 0 \leq t \leq T \\ u(x,0) = u_0(x) & for\ x \in \overline{\Omega}, \end{cases}$$

for general (regular) boundary data φ and initial data u_0, with $\varphi(x,0) = u_0(x)$.