Linear Algebra/Advanced Calculus Basic Exam
Winter 2006

Do 7 of the following 9 problems.

Passing standard: For Master's level, 60% with three questions essentially correct (including at least one from each part). For Ph.D. level, 75% with two questions from each part essentially complete.

Part I: Linear algebra

1. Let V be a vector space over a field k and let W_1 and W_2 be subspaces of V. Prove that if $W_1 \cup W_2$ is a subspace of V, then either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

2. Let A be an $n \times n$ real matrix and let $v \in \mathbf{R}^n$ be a vector. Suppose that there is an integer $k \geq 1$ such that $A^k v \neq 0$ and $A^{k+1} v = 0$. Prove that

$$\{v, Av, A^2v, \ldots, A^k v\}$$

is linearly independent.

3. (a) Let A and B be nilpotent 6×6 complex matrices with the same rank and minimal polynomial. Prove that A and B are similar.

   (b) Give two nilpotent 7×7 complex matrices with the same rank and minimal polynomial which are not similar.

4. Find the Jordan canonical form of the matrix

$$\begin{bmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{bmatrix}.$$

(Hint: the characteristic polynomial of this matrix is $x^3 - 3x^2 + 4$.)

Part II: Advanced calculus

1. Compute the line integral

$$\int_C (\cos x + 2xy + y)dx + (ye^y + x^2)dy$$

where C is the upper unit semicircle, directed from $(1,0)$ to $(-1,0)$.

2. Let $\{f_n(t)\}_{n\geq 1}$ be a sequence of continuous functions from $[0,\infty)$ to $\mathbf{R}$ which converge to a function $g(t)$ in the sup norm. Let

$$F_n(x) = \int_0^x f_n(t)dt$$

$$G(x) = \int_0^x g(t)dt.$$

Prove that $\{F_n(x)\}_{n\geq 1}$ converges to $G(x)$ pointwise. Does $\{F_n(x)\}_{n\geq 1}$ converge to $G(x)$ in the sup norm?

3. Let $\{(-1)^j a_j\}_{j\geq 1}$ be an alternating series. (In particular in this notation we have $a_j \geq 0$.)

   (a) State the alternating series test for the convergence of the sum

   $$\sum_{j=1}^{\infty} (-1)^j a_j.$$

   (b) Assume that $\sum_{j=1}^{\infty} (-1)^j a_j$ converges; we write S for its value. Let $S_n = \sum_{j=0}^{n} (-1)^j a_j$ be the n^{th} partial sum. Prove that $|S - S_n| \leq a_{n+1}$.

   (c) Give a bound on the difference between $\cos x$ and

   $$1 - \frac{x^2}{2} + \frac{x^4}{24}$$

   on the interval $[-1,1]$.

4. Let $\{a_n\}_{n\geq 1}$ be a bounded monotonic increasing sequence. Define

$$b_n = \frac{a_1 + a_2 + \cdots a_n}{n}.$$

Prove that the sequence $\{b_n\}_{n\geq 1}$ is a bounded monotonic increasing sequence and that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

5. Let $f(x,y,z)$ be a function with continuous second-order partial derivatives. Prove that

$$\int_C (f\,\nabla f)\cdot d\mathbf{r} = 0$$

for any smooth simple closed curve C in $\mathbf{R}^3$. (You may use the fact that any such C is the boundary of a smooth surface S.)