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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space with $\mu(X) > 0$ and let f be a nonnegative, Borel-measurable function mapping X into $[0, \infty)$. For $A \in \mathcal{M}$, define

$$\lambda(A) = \int_A f \, d\mu \, .$$

(a) Prove that λ is a measure on $\mathcal{M}$.

(b) Assume that f is also bounded. Prove that λ is a *finite* measure.

(c) Assume that $f \neq 0$ a.e. Prove that there exists $A \in \mathcal{M}$ such that $\lambda(A) > 0$. This proves that the measure λ is nontrivial.

(d) Let g be any nonnegative, Borel-measurable function mapping X into $[0, \infty)$. Prove that

$$\int_X g \, d\lambda = \int_X fg \, d\mu \, .$$

2. Let $\mathcal{H}$ denote the Hilbert space $L^2[0,1]$ with respect to Lebesgue measure. Let $\mathcal{D}$ denote the span of the functions 1 and x in $\mathcal{H}$.

   (a) Determine an orthonormal basis of $\mathcal{D}$.

   (b) Is the orthonormal basis that you found in part (a) unique? If not, indicate another orthonormal basis.

   (c) Define $f(x) = x^2$ for $x \in [0,1]$. Find an element $g \in \mathcal{D}$ satisfying

$$\|f - g\| \;=\; \inf_{h \in D} \|f - h\|\,.$$

   Explain all your steps and state carefully any theorems about Hilbert space that you need.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n, n \in I\!N\}$ a sequence of Borel-measurable functions mapping X into $I\!R$.

(a) Define the concept $f_n \to 0$ a.e. as $n \to \infty$.

(b) We say that $f_n \to 0$ in measure as $n \to \infty$ if for all $\delta > 0$, $\lim_{n\to\infty} \mu(A_n(\delta)) = 0$, where $A_n(\delta)$ is a set defined in terms of f_n and δ. Indicate the definition of $A_n(\delta)$.

(c) Let $B = \{x \in X : \lim_{n\to\infty} f_n(x) = 0\}$.

(i) Express B^c, the complement of B, as a countable union of a countable intersection of a countable union of the sets $A_n(\delta)$ in part (b) for appropriate choices of δ.

(ii) Assume that for any $\delta > 0$ there exists $c < \infty$ such that for all $n \in I\!N$, $\mu(A_n(\delta)) \leq c/n^2$. Prove that $\mu(B^c) = 0$. What does this say about the convergence of f_n to 0 as $n \to \infty$.

4. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $\{f_n, n \in I\!N\}$ be a sequence of nonnegative, Borel-measurable functions mapping X into $I\!R$ and satisfying $f_n \to 0$ in L^1 as $n \to \infty$.

   (a) Prove that $\sqrt{f_n} \to 0$ in L^1 as $n \to \infty$. (**Hint.** For each n split X into $\{x \in X : f_n \geq \varepsilon\}$ and $\{x \in \mathcal{X} : f_n < \varepsilon\}$.)

   (b) Give an example to show that f_n^2 need not converge to 0 in L^1 as $n \to \infty$.

5. Let $(X, \mathcal{M})$ be a measurable space; let μ be a finite, positive measure on this space; and let ν be a finite, signed measure on this space. Denote by $|\nu|$ the positive measure that is the total variation of ν. Prove that the following statements (a) and (b) are equivalent:

   (a) For all $E \in \mathcal{M}$, $|\nu(E)| \leq \mu(E)$.

   (b) $\nu \ll \mu$ and $\left|\frac{d\nu}{d\mu}(x)\right| \leq 1$ for μ-a.e. $x \in X$.

6. Let a and b be real numbers satisfying $-\infty < a < b < \infty$. We denote by $BV([a, b])$ the set of all functions F that map $[a, b]$ into $I\!R$ and that are of bounded variation on $[a, b]$.

(a) Define the concept that F is of bounded variation on $[a, b]$.

(b) Let F be a differentiable function mapping $I\!R$ into $I\!R$. Prove that if the derivative F' is bounded on $I\!R$, then $F \in BV([a, b])$ for any $-\infty < a < b < \infty$. (**Hint.** Use the mean value theorem.)

(c) Define

$$G(x) = x^2 \sin(1/x) \text{ and } H(x) = x^2 \sin(1/x^2)$$

for $x \neq 0$ and $G(0) = H(0) = 0$. Prove the following.

(i) G and H are differentiable at all $x \in I\!R$, including at $x = 0$.

(ii) $G \in BV([-1, 1])$, but $H \notin BV([-1, 1])$.

7. (a) Lebesgue measure m on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ has the property that for any Borel set E in $\mathbb{R}$ and any real number s,

$$m(E+s) = m(E)\,.$$

What is this property of Lebesgue measure called? You need not prove this property.

(b) Let g be an arbitrary bounded, Borel-measurable function mapping $\mathbb{R}$ into $\mathbb{R}$ and A an arbitrary bounded interval in $\mathbb{R}$. Prove that $\int_A |g| dm < \infty$.

(c) Let g be an arbitrary bounded, Borel-measurable function mapping $\mathbb{R}$ into $\mathbb{R}$; let $[\alpha, \beta]$ be an arbitrary bounded, closed interval in $\mathbb{R}$; and let t an arbitrary real number. Using the property of m given in part (a), prove that

$$\int_{[\alpha,\beta]} g(x+t)\, dm(x) = \int_{[\alpha+t,\beta+t]} g(x)\, dm(x)\,.$$

8. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{D}$ be a nontrivial closed subspace of $\mathcal{H}$. Define

$$\mathcal{D}^{\perp} = \{x \in \mathcal{H} : \langle x, y \rangle = 0 \text{ for all } y \in \mathcal{D}\}.$$

According to a basic theorem about Hilbert space, which you need not prove, there exists a unique linear operator P mapping $\mathcal{H}$ into $\mathcal{H}$ and satisfying the following: for all $x \in \mathcal{H}$, $Px \in \mathcal{D}$ and $x - Px \in \mathcal{D}^{\perp}$. P is called the orthogonal projection onto $\mathcal{D}$.

(a) Prove the following: (i) $\mathcal{D} \cap \mathcal{D}^{\perp} = \{0\}$; (ii) $Px = x$ for all $x \in \mathcal{D}$; (iii) $Px = 0$ for all $x \in \mathcal{D}^{\perp}$.

(b) Prove that $\|P\| = 1$, where $\| \cdot \|$ denotes the operator norm.

(c) According to another basic theorem about Hilbert space, which you need not prove, there exists a unique linear operator $P^* : \mathcal{H} \to \mathcal{H}$ with the property that $\langle Px, y \rangle = \langle x, P^* y \rangle$ for all $x, y \in \mathcal{H}$. Using this property of P^*, prove that $P^* = P$. (**Hint.** You may use without proof the fact that $(\mathcal{D}^{\perp})^{\perp} = \mathcal{D}$.)