Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

January 19, 2005

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Show that a continuous map $S^1 \to \mathbb{R}$ cannot be either injective or surjective.

(2) Let X be a topological space.
  (a) Write careful definitions of the statements "X is connected", "X is path connected" and "X is locally path-connected".
  (b) Show directly from the definitions that a connected and locally path-connected space is path-connected.

(3) Let X be a topological space.
  (a) Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X \mid x \in X\}$ is closed in $X \times X$.
  (b) Suppose that X is Hausdorff. Let A be a dense subset of a space Y. Show that if $f, g \colon Y \to X$ are continuous functions that agree on A (i.e. $f|_A = g|_A$), then $f = g$.

(4) Let X and Y be spaces, and suppose that X is compact. Show directly from the definitions that the projection $\pi \colon X \times Y \to Y$ is a closed map.

(5) Let X be a metric space, and $A \subset X$ a subspace. Recall that X/A denotes the quotient space of X where all the points of A have been identified.
  (a) Show that X/A is Hausdorff if and only if A is closed.
  (b) If $X = \mathbb{R}^2$ and A is the closed unit ball, show that X/A is homeomorphic to X.

(6) Define a sequence of functions $\{f_n\}$, $f_n \colon \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ e^{nx} & \text{if } x < 0. \end{cases}$$

Determine whether or not the sequence converges in the point-open, uniform, and compact open topologies (the point-open topology is the same as the product topology, where the space of functions $\mathbb{R} \to \mathbb{R}$ is considered as a product of uncountably many copies of $\mathbb{R}$).

(7) Let X be a metric space, and let $B([0, 1], X)$ denote the set of bounded functions $[0, 1] \to X$, endowed with the sup norm metric. Show that $B([0, 1], X)$ is complete if and only if X is complete.