Department of Mathematics and Statistics
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# Basic Exam: Topology

January 23, 2004

**Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.**
**Passing standard:** For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Let $X = \mathbb{R}$, with the topology whose basic open sets are $(-\infty, a)$ for $a \in \mathbb{R}$. Let functions $f_1$ and $f_2$ from $\mathbb{R}$ to $X \times X$ be defined by $f_1(x) = (x, x)$ and $f_2(x) = (x, -x)$. For $i = 1, 2$, find the topology on $\mathbb{R}$ which makes $f_i$ a homeomorphism with the subspace $f_i(\mathbb{R})$, thought of as a subspace of $X \times X$.

(2) Given $(a_1, a_2, \dots) \in \mathbb{R}^\omega$, define a function $f\colon \mathbb{R} \to \mathbb{R}^\omega$ by

$$f(t) = (a_1 t, a_2 t, \dots).$$

Give necessary and sufficient conditions for $f$ to be continuous if $\mathbb{R}^\omega$ is given the box topology or the product topology.

(3) (a) Let $X = (0, 1)$. Prove that the one-point compactification $\hat{X}$ of $X$ is homeomorphic to the circle $S^1$.
(b) Find a topological space $Y$ and a continuous function $X \to Y$ which does not extend to a continuous function $\hat{X} \to Y$.

(4) Let $A$ and $B$ be proper subsets of spaces $X$ and $Y$, respectively. If $X$ and $Y$ are both connected, show that $(X \times Y) - (A \times B)$ is connected. (Hint: try looking at the case $X = Y = [0, 1]$, $A = B = (0, 1)$).

(5) Let $(X, d)$ be a metric space, and let $A$ be a nonempty subset of $X$. For each $x \in X$, define $d(x, A) = \inf\{d(x, a) \mid a \in A\}$.
(a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.
(b) Show that if $A$ is compact, then $d(x, A) = d(x, a)$ for some $a \in A$.

(6) Let $X$ be the quotient of $[0, \infty)$ obtained by identifying $n$ with $1/n$ for all integers $n > 1$.
(a) Show that the quotient map $q\colon [0, \infty) \to X$ is not an open map.
(b) Show that $X$ is not compact.

(7) Let $X$ be the set of all functions $[0, 1] \to [0, 1]$, endowed with the sup-metric: $d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. Let

$$F = \{f \in X \mid f([0, 1]) \text{ is finite}\}, \text{and}$$

$$C = \{f \in X \mid f \text{ is continuous}\}.$$

Show that $C$ is contained in the closure of $F$.