DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - NUMERICS
January 20, 2004

Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correct.
Ph.D.: 75% with at least three substantially correct.

1. Determine values of α in the matrix below

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 5 & 8 \\ 4 & 8 & \alpha \end{pmatrix}$$

   for which the matrix is positive definite.

2. (a) Write down the Jacobi and the Gauss-Siedel methods for the system $Ax = b$ where $A =$

$$\begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix}$$

   and $b =$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

   (b) Determine if the Gauss-Siedel method will converge for any initial guess.

3. The *false position method* is a variation on the bisection method, where the next test point c of the bracketing interval $[a, b]$ is the point where the line through the points $(a, f(a))$ and $(b, f(b))$ crosses the x-axis (rather than $c = \frac{a+b}{2}$ for bisection).

(a) Write down the algorithm for a general function in detail.

(b) Compare this method to the secant method. Give an example showing that this is different from the secant method (no need to give f , a picture with a few words should do).

4. We want to approximate

$$\int_0^2 f(x)x^2 dx$$

by a rule of the form

$$af(b).$$

Find a and b so that the method is exact for as many polynomials as possible. Also find the error term.

5. For the trapezoidal rule (denoted by I^T) for evaluating

$$I = \int_a^b f(x)\, dx,$$

we have the asymptotic error formula

$$I^T = I + C_1 h^2 + O(h^4),$$

and for the midpoint formula I^M, we have

$$I^M = I + C_2 h^2 + O(h^4),$$

where C_1 and C_2 are constants. Using these results, obtain a new numerical integration formula $\widetilde{I}$ combining I^T and I^M with a higher order of convergence. Write out the weights to the new formula $\widetilde{I}$.

6. Find the polynomial $q(x) = a_0 + a_1 x$ of degree one or less which approximates e^x best in the sense that it minimizes the error

$$\int_0^1 (e^x - q(x))^2 \, dx.$$

7. Consider the ODE initial-value problem

$$\frac{dy}{dx}(x) = f(x, y(x)),$$

with initial data $y(x_0) = y_0$. We would like to solve this initial value problem at points $x_n = nh, n = 0, ..., N$ where $h = x_n - x_{n-1}$ for all n. Find the highest order method in the class

$$y_{n+1} = y_n + h[b_1 f(x_n, y_n) + b_2 f(x_{n-1}, y_{n-1})].$$

i.e., find b_1 and b_2 for the above method which gives the highest order local truncation error. State the order of the method obtained.