Department of Mathematics and Statistics
University of Massachusetts

Basic Exam - Complex Analysis

January 22, 2004

Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master's level it is necessary to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are required for passing at the Ph.D. level.

Note: All answers should be justified.
counterclockwise.

1. For the function $f(z) = e^{\frac{z+1}{z-1}}$ show that

   (a) $|f(z)| < 1$ when $|z| < 1$.

   (b) $|f(z)| = 1$ when $|z| = 1$ and $z \neq 1$.

   (c) The left limit at 1 along the real axis is zero:

   $$lim_{x \to 1^-}\ f(x) =\ 0.$$

2. Evaluate the integral

   $$\int_0^{+\infty} \frac{\ln(x)}{x^2+1}\ dx.$$

   Hint: Consider a contour consisting of two semi-circles, centered at the origin, and two line-segments along the x-axis. Include a proof, that the contour you chose can be used to evaluate the integral.

3. Compute the following integral:

   $$\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+x+1} dx.$$

   Justify all your steps!

4. Evaluate the integral

   $$\int_0^{\pi} \frac{\cos^2(\theta)}{2+\cos(\theta)}\ d\theta.$$

   Justify all steps.

5. (a) Let ρ be a real number satisfying $0 < \rho < 1$. Find a one-to-one conformal map $f(z)$ from the unit disc to itself, such that $f(0) = -\rho$ and $f(1/2) = \rho$.

   (b) Show that $f(z)$ sends the region between two circles (with different centers)

$$B = \{|z| < 1 \text{ and } |z - 1/4| > 1/4\}, \text{ to the annulus } A = \{\rho < |z| < 1\},$$

   and that the restriction of f to a map $f : B \to A$ is an isomorphism.

6. (a) Consider the lens region L which is the intersection of two discs bounded by the circles C_1 and C_2, such that C_1 passes through $-1, (\sqrt{2} - 1)i, 1$, and C_2 passes through $-1, (1 - \sqrt{2})i, 1$. Find a fractional linear transform $f(z)$ that maps L isomorphically onto the first quadrant, so that $f(-1) = 0$, $f(1) = \infty$ and $f((1 - \sqrt{2})i) = 1$. (Check that f satisfies all properties!)

   (b) Find a function u, harmonic on L and such that $u = 1$ on the upper boundary of L and $u = 0$ on the lower boundary.

7. Find all Laurent series of

$$f(z) = \frac{z}{(z-1)(z+1)(z+i)}$$

   around $z = 1$, indicating the region of convergence for each series. Express each Laurent series of f as a single series, rather than a product of series.

8. Let $f(z)$ be an analytic function on the disc $A = \{|z| < 3\}$, such that $|f(z)| \geq 2$ when $|z| \geq 2$, while $|f(z)| \leq 1$ when $|z| \leq 1$. Show that $f(z)$ has a zero in A.

9. Show that the function $f(z) = z - 3 + 2e^{-z}$ has precisely one zero in the right half plane $Re(z) > 0$.
   (*Hint*: Consider bounded regions such as the ones given by rectangular contours with vertices at $(0, \pm R)$ and $(R, \pm R)$, for large $R > 0$.)

10. State and prove the Casorati-Weierstrass Theorem regarding the behavior of a function holomorphic except for an essential singularity at a point z_0 in the complex plane.